pen Tournament

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Contest Solutions (Europe/Africa)

Middle School Division

Saturday, March 26, 2022

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1. Assuming that 1 meter is equal to 100 centimeters, 1 yard is equal to 3 feet, 1 foot is equal to 12 inches, and 40 inches is equal to 1 meter, how many centimeters are equal to one yard?

   Answer: $\boxed{90}$

   One yard is equal to 3 feet, which is equal to $12 \cdot 3 = 36$ inches. 36 inches is $\frac{9}{10}$ of 40 inches, so it is $\frac{9}{10}$, or 90 percent, of one meter, meaning that it is equal to $\boxed{90}$ centimeters.

2. Blanche writes, for each $1 \leq k \leq 26$, k copies of the k^{th} letter of the alphabet in a row, so that her string begins ABBCCCDDDD $\cdots$; and ends with 26 Z's. What is the middle letter in her string?

   (a) M
   (b) P
   (c) R
   (d) T
   (e) none of the above

   Answer: $\boxed{E}$

   There are a total of $1+2+3+\cdots+26 = 351$ letters in Blanche's string; we want the 176^{th}. That is, we want the smallest positive integer n such that $1+2+3+\cdots+n \geq 176$. Since $1+2+3+\cdots+18 = 171$ and $1+2+3+\cdots+19 = 190$, we know that $n = 19$. The 19^{th} letter of the alphabet is $\boxed{S}$.

3. The circumference of a $150°$ sector of a circle with integer radius is $15\pi + k$ for some integer k. What is k?

   (a) 18
   (b) 24
   (c) 36
   (d) 60
   (e) none of the above

   Answer: $\boxed{C}$

   As a $150°$ sector of a circle has arc length $\frac{150}{360} = \frac{5}{12}$ times that of the entire circle, 15π is $\frac{5}{12}$ the circumference of the entire circle, or 36π. As the circumference of a circle with radius r is $2\pi r$, this implies that $r = 18$; because the sector consists of an arc length and two radii, we have that $k = 2r = \boxed{36}$.

4. The sum of the square roots of three distinct positive integers a, b, and c summing to 35 is an integer. Compute the product abc.

   Answer: $\boxed{225}$

   All of a, b, and c must be perfect squares, so they are each one of 1, 4, 9, 16, or 25. In order for a, b, and c to sum to 35, they must be (some permutation of) 1, 9, and 25, which can be seen by inspection. Hence, $abc = \boxed{225}$.

5. How many positive integers between 1 and 100, inclusive, are the positive difference between two numbers of the form $N^2 - N + 1$ for some positive integer N?

   Answer: $\boxed{50}$

   All even positive integers can be expressed as such a difference; consider $((n+1)^2 - (n+1) + 1) - (n^2 - n + 1) = 2n$. Note, however, that $N^2 - N$ is always even, so $N^2 - N + 1$ is always odd. An odd number cannot be the positive difference between two odd numbers, so we have found all desired integers. There are $\boxed{50}$ even integers between 1 and 100, inclusive.

6. Sixty times a positive integer leaves a remainder of 58 when divided by 119. Compute the smallest possible value of this positive integer.

   Answer: $\boxed{116}$

   Call the positive integer n; then if $60n \equiv 58 \bmod 119$, $120n \equiv 116 \bmod 119$, or $n \equiv 116 \bmod 119$. This makes the smallest possible value of n equal to $\boxed{116}$.

7. Triangle ABC has $AB = AC = 10$ and $BC = 12$. Point D lies on $\overline{BC}$ between B and C with $BD = 10$. Compute AD^2.

   Answer: $\boxed{80}$

   By dropping the altitude from A to the midpoint M of $\overline{BC}$, we observe that $AM = 8$ and $BM = MC = 6$. Thus, $MD = 10 - 6 = 4$, and by the Pythagorean theorem, $AD^2 = 8^2 + 4^2 = \boxed{80}$.

8. Chester flips 4 fair coins, and Rhiannon flips 6 fair coins. What is the probability that Rhiannon flips more heads than Chester? Express your answer as a common fraction.

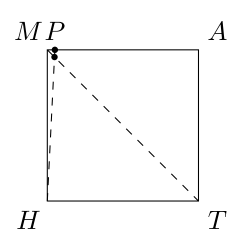
   Answer: $\boxed{\frac{319}{512}}$

   It's well-known (by symmetry) that the probability of Rhiannon flipping more heads than Chester if Chester flips c coins and Rhiannon flips $c + 1$ coins is $\frac{1}{2}$. (The sketch of the argument uses the symmetry of the choose function: if 0 of Chester's flips are heads, all Rhiannon needs to do is flip one head, i.e. not flip all tails; but if all of Chester's flips are heads, Rhiannon must flip all heads. Likewise, in general, if Chester flips $h < c$ heads, Rhiannon must flip at least $h + 1$ heads; but if Chester were to flip $c - h$ heads, Rhiannon would need to *not* flip at least $c - h$ tails.) Given that this event with probability $\frac{1}{2}$ occurs (with $c = 4$), the sixth coin is irrelevant: the problem condition is already satisfied. Otherwise, Rhiannon's sixth coin can only push her over Chester's total if Chester and Rhiannon are tied with Chester having flipped 4 coins and Rhiannon having flipped 5 coins. The probability this occurs is $\frac{1}{2}$ of the sum of probabilities $\frac{\binom{4}{0}\binom{5}{0}+\binom{4}{1}\binom{5}{1}+\binom{4}{2}\binom{5}{2}+\binom{4}{3}\binom{5}{3}+\binom{4}{4}\binom{5}{4}}{2^9} = \frac{1+20+60+40+5}{512} = \frac{63}{256}$, hence $\frac{63}{512}$. Adding this to the probability of $\frac{1}{2}$ that Rhiannon has *more* heads than Chester after she flips 5 coins and Chester flips 4 gives our final probability of $\boxed{\frac{319}{512}}$.

9. Square $MATH$ has side length 2. Point P lies on $\overline{MA}$ such that the area of quadrilateral $PATH$ is 3.9. Compute the area of overlap between quadrilateral $PATH$ and triangle MTH. Express your answer as a common fraction.

   Answer: $\boxed{\frac{40}{21}}$

   We can draw the following diagram:



Call Q the intersection point of $\overline{MT}$ and $\overline{PH}$. Since we know that $\frac{MP}{MA} = \frac{1}{20}$, we have $\frac{MQ}{QT} = \frac{1}{20}$ by similarity, whence $QT = \frac{20}{21}MT$, and the area of $\triangle QTH$ is $\frac{20}{21}$ times that of $\triangle MTH$, which is just 2. Thus, the desired area is $\boxed{\dfrac{40}{21}}$.

10. Compute the sum of all integers n such that

$$\frac{\sqrt{n}+5}{\sqrt{n}+1}$$

is an integer.

Answer: $\boxed{10}$

The given quantity can be rewritten as $1 + \frac{4}{\sqrt{n}+1}$, so we want $\frac{4}{\sqrt{n}+1}$ to be an integer. As $\sqrt{n} + 1 \geq 1$, this requires that $\sqrt{n} + 1 = 1, 2, 4$, so that $n = 0, 1, 9$ respectively, which sum to $\boxed{10}$.

11. An isosceles triangle has a base length of 10 and two side lengths of 13. A point inside the triangle is equidistant with common distance d from all three vertices of the triangle. Then d can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: $\boxed{193}$

Call the triangle ABC, with $AB = AC = 13$ and $BC = 10$. Let P be the point inside $\triangle ABC$ with $PA = PB = PC$. Then P must be at a horizontal distance of 5 from both B and C; suppose its distance from $\overline{BC}$ is h. Then $d^2 = 25 + h^2$, and is also equal to $(12 - h)^2$, since we can split $\triangle ABC$ down its A-altitude into two 5-12-13 right triangles (where the altitude length is 12). Thus, $25 + h^2 = 144 - 24h + h^2$, from which we get $h = \frac{119}{24}$ and the common distance d as $12 - h = 12 - \frac{119}{24} = \frac{169}{24}$, so $m + n = 169 + 24 = \boxed{193}$.

12. A positive integer is called *stable* if none of its digits are greater than the cube of the smallest digit. Compute the number of stable positive integers less than or equal to 1000.

Answer: $\boxed{543}$

Say the smallest digit is 0; then all digits must be 0, which is a contradiction. If the smallest digit is 1, all digits must be 1, which gives 1, 11, and 111 as stable numbers. If the smallest digit is 2, all digits must lie between 2 and 8, inclusive, so we have 2, 22 through 28, and the numbers from 32 through 82 ending in 2 as two-digit stable numbers, of which there are 14.

For three-digit stable numbers with a smallest digit of 2, if 2 is the hundreds digit, we have $7^2 = 49$ stable numbers: namely, numbers of the form $2xy$ where $2 \leq x, y \leq 8$. If 2 is the tens digit, we have 323 through 328, 423 through 428, 523 through 528, 623 through 628, 723 through 728, and 823 through 828, for another $6^2 = 36$ stable numbers. If 2 is the units digit, we similarly get another $6^2 = 36$ stable numbers. We also have 6 numbers of the form $x22$ (for $3 \leq x \leq 8$) which have not yet been counted.

Finally, if the smallest digit is at least 3, there are no other restrictions on the digits, so 3 through 9, the $7^2 = 49$ two-digit positive integers with digits from 3-9, and the $7^3 = 343$ three-digit positive integers with digits from 3-9. Altogether, we have $3 + 14 + 49 + 36 + 36 + 6 + 7 + 49 + 343 = \boxed{543}$ stable numbers less than or equal to 1000.

13. Given that

$$7 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{673} < 8,$$

compute the largest integer not exceeding

$$\frac{3}{1+2}+\frac{6}{4+5}+\frac{9}{7+8}+\frac{12}{10+11}+\cdots+\frac{2022}{2020+2021}.$$

Answer: $\boxed{339}$

This is equal to

$$1+\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\cdots+\frac{674}{1347}=1+\left(\frac{1}{2}+\frac{1}{6}\right)+\left(\frac{1}{2}+\frac{1}{10}\right)+\left(\frac{1}{2}+\frac{1}{14}\right)+\cdots+\left(\frac{1}{2}+\frac{1}{2694}\right).$$

Noting that there are 674 terms in the sum, all of the $\frac{1}{2}$'s sum to $\frac{1}{2}\cdot 674=337$. Along with the 1, we have a sum of 338 up to this point.

The remaining terms sum to

$$\frac{1}{6}+\frac{1}{10}+\frac{1}{14}+\cdots+\frac{1}{2694}=\frac{1}{1\cdot 4+2}+\frac{1}{2\cdot 4+2}+\frac{1}{3\cdot 4+2}+\cdots+\frac{1}{673\cdot 4+2}.$$

We can bound this sum S strictly between

$$\frac{1}{1\cdot 4}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{673\cdot 4}$$

and

$$\frac{1}{2\cdot 4}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 4}+\cdots+\frac{1}{674\cdot 4}.$$

The first sum is $\frac{1}{4}$ of H_{673}, where H_n is defined as the sum $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and the second sum is $\frac{1}{4}$ of the quantity $H_{674}-1$. By the problem statement, we know that $7<H_{673}<8$, so $\frac{7}{4}<\frac{1}{4}H_{673}<2$. Similarly, $6+\frac{1}{674}<H_{674}-1<7+\frac{1}{674}$, so $\frac{3}{2}+\frac{1}{4\cdot 674}<\frac{1}{4}(H_{674}-1)<\frac{7}{4}+\frac{1}{4\cdot 674}<2$. In other words, we know that $1<S<2$, so $338+S$ is between 339 and 340, and the largest integer not exceeding the desired sum is $\boxed{339}$.

In general, we can observe that

$$H_n<1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\cdots+\frac{1}{8}\right)+\cdots$$

(with eight $\frac{1}{8}$s), with each block summing to 1, which shows that $H_{2^n-1}<n$. Since $2^9-1<673<2^{10}-1$, we know that $H_{673}<10$. At the same time,

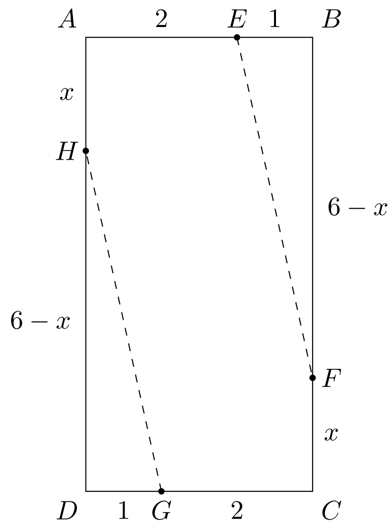
$$H_n>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots,$$

meaning that $H_{2^n}>2+\frac{n}{2}$ also.

14. Let rectangle $ABCD$ have $AB=3$ and $BC=6$. Points E, F, G, and H lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that $EB=GD=1$ and $EF=GH$. If the perimeter of $AEFCGH$ is at most 90 percent of the perimeter of $ABCD$, compute the maximum possible length of $\overline{FC}$.

Answer: $\boxed{\frac{21}{20}}$

We draw the following diagram.



We have $AE = GC = 2$; for $FC = x$, we have $FB = 6 - x$, so by the Pythagorean theorem, and by symmetry, $EF = GH = \sqrt{37 - 12x + x^2}$. Thus, as the perimeter of $ABCD$ is 18, we have $2 + \sqrt{37 - 12x + x^2} + x \le \frac{81}{10}$, or $\sqrt{37 - 12x + x^2} \le \frac{61}{10} - x$. Squaring both sides yields $37 - 12x + x^2 \le \frac{3721}{100} - \frac{61}{5}x + x^2$, or $\frac{21}{100} - \frac{1}{5}x \ge 0$, i.e. $x \le \boxed{\frac{21}{20}}$.

15. The *digital root* of a positive integer is the result of repeatedly summing the digits of that integer until a single integer from 1 to 9, inclusive, is obtained. For example, the digital root of 2022 is 6, the digital root of 1234567 is 1 (since the sum of digits is 28, $2 + 8 = 10$, and $1 + 0 = 1$), and the digital root of 36 is 9. Compute the sum of the digital roots of all the positive integers from 1 to 2022, inclusive.

Answer: $\boxed{10101}$

We claim the digital root of n is congruent to $n \bmod 9$ and is an integer in $[1, 9]$. It suffices to show that the digit sum is congruent to $n \bmod 9$, since the digital root is obtained from repeatedly iterating the digit sum operation. Recall that any base-10 positive integer n can be written as a sum of powers of 10 multiplied with each of the corresponding digits, and since $10^k \equiv 1 \bmod 9$ for all non-negative integers k, the sum of digits is congruent to the sum of 10^k times the digits over all $0 \le k \le m$, where m is the number of digits of n.

To compute the sum of the digital roots of the integers from 1 to 2022, note that we can sum the remainders when each of them is divided by 9, except for the multiples of 9, which have digital roots of 9 rather than 0. The sum of the digital roots in each block of nine is $1 + 2 + 3 + \cdots + 9 = 45$. The largest multiple of nine less than or equal to 2022 is $2016 = 9 \cdot 224$, so we have a sum of digital roots of $45 \cdot 224 = 10080$ up to, and including, 2016. Finally, the sum of the digital roots of 2017 through 2022 is $1 + 2 + 3 + 4 + 5 + 6 = 21$, so we get a final sum of $10080 + 21 = \boxed{10101}$.

16. A set consists of sixteen distinct positive integers which sum to 139. When one of these sets is chosen uniformly at random, compute the expected value of its largest element.

Answer: $\boxed{18}$

Observe that $1 + 2 + 3 + \cdots + 16 = 136$ and that $1 + 2 + 3 + \cdots + 17 = 153$, so the set may be one of $\{1, 2, 3, \cdots, 15, 19\}$, $\{1, 2, 3, \cdots, 14, 16, 18\}$, or $\{1, 2, 3, \cdots, 13, 15, 16, 17\}$. If we tried setting the maximal element to 20 or larger, the smallest 15 elements, which would need to sum to at least $1 + 2 + 3 + \cdots + 15 = 120$, would force the sum of the elements in such a set to be at least 140, which is a contradiction. The expected value of the largest element is then $\frac{19+18+17}{3} = \boxed{18}$.

17. For each positive integer n, define
$$f(n) = \frac{\sum_{i=1}^{n}(i+i^2)}{\sum_{i=1}^{n} i^3}.$$
Compute the smallest positive integer n for which $f(n) \le \frac{1}{10}$.

Answer: $\boxed{15}$

The numerator simplifies to
$$\left(\sum_{i=1}^{n} i\right) + \left(\sum_{i=1}^{n} i^2\right) = \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{3},$$
while the denominator is $\left(\frac{n(n+1)}{2}\right)^2$. Thus, $f(n)$ simplifies to $\frac{4}{3} \cdot \frac{n+2}{n^2+n}$. In order for this to be at most $\frac{1}{10}$, we must have $\frac{n+2}{n^2+n} \le \frac{3}{40}$, or $40n + 80 \le 3n^2 + 3n \implies 3n^2 - 37n - 80 \ge 0$. This occurs when $n \ge \boxed{15}$ (by the quadratic formula).

18. Let $ABCD$ be a rectangle with $AB = 2$ and $BC = 1$. Suppose that E and F are points on $\overline{AB}$ and $\overline{CD}$, respectively, lying on the same side of $\overline{BC}$, such that $AE \cdot CF = 1$ and $EF = \frac{5}{4}$. The largest possible length of CF can be written in the form $\frac{p+\sqrt{q}}{r}$, where p, q, and r are positive integers with q not divisible by the square of a prime. Compute $p + q + r$.

Answer: $\boxed{204}$

By the Pythagorean theorem, the horizontal distance between E and F is $\sqrt{\left(\frac{5}{4}\right)^2 - 1^2} = \frac{3}{4}$. Letting $AE = x$ (with $BE = x + 2$) and $CF = x + \frac{11}{4}$, we solve the equation $x\left(x + \frac{11}{4}\right) = 1$, and get that $4x^2 + 11x - 4 = 0$, or $x = \frac{11 \pm \sqrt{185}}{8}$. Taking the positive value, we get that $p+q+r = 11+185+8 = \boxed{204}$.

19. Let n be a positive integer. A permutation of $(a_1, a_2, a_3, \cdots, a_{2n})$ is called *rightweight* if $2(a_1+a_2+a_3+\cdots+a_n) \le a_{n+1}+a_{n+2}+a_{n+3}+\cdots+a_{2n}$. Compute the number of permutations of $(1, 2, 3, 4, 5, 6, 7, 8)$ that are rightweight.

Answer: $\boxed{2304}$

The sum of all elements is 36, so we require $a_1 + a_2 + a_3 + a_4 \le 12$. The possible 4-tuples (a_1, a_2, a_3, a_4) are the permutations of $(1, 2, 3, 4)$, $(1, 2, 3, 5)$, $(1, 2, 3, 6)$, and $(1, 2, 4, 5)$, giving $4!^2 \cdot 4 = \boxed{2304}$ rightweight permutations.

20. Let $\tau(n)$ denote the number of positive integer divisors of n. Compute the number of positive integers $n \le 100$ satisfying $\tau(n) + \tau(2n) = 18$.

Answer: $\boxed{14}$

In the first case, n is odd, in which case $\tau(2n) = 2\tau(n)$ and $\tau(n) = 6$. Noting that the number of divisors of a positive integer is equal to the products of the terms of the form $1 + k_i$, where prime factor p_i has exponent k_i, we observe that n can be of the forms p^5 or p^2q for primes $p \ne q$. In the first case, we have no values of n (since $n = 2^5 = 32$ is even, and for $p \ge 3$, we get $n > 100$); in the latter, we may have $n = 3^2 \cdot 5 = 45$, $3^2 \cdot 7 = 63$, $3^2 \cdot 11 = 99$, or $5^2 \cdot 3 = 75$.

In the second case, n has a power of 2^k in its prime factorization for some integer $k \ge 1$, implying that $\tau(2n) = \frac{k+2}{k+1}\tau(n)$. This means that $\tau(n)\left(\frac{2k+3}{k+1}\right) = 18$, so $2k+3$ must divide $18(k+1) = 18k + 18$. By the Euclidean algorithm, $2k + 3 \mid (18k + 18) \iff (2k + 3) \mid -9$, so $k = 3$ is the only possible value, implying that $\tau(n) = 8$. Then $n = 2^7$, $2^3 \cdot p$ for some prime p, $2p^3$, or $2pq$ for distinct primes p and q.

As $2^7 > 100$, we discard this subcase. For $n = 2^3 \cdot p$, we have $n \in \{16, 24, 40, 56, 88\}$, and for $n = 2pq$, we have $n \in \{30, 42, 66, 78, 70\}$. This gives 10 additional values of n. Hence, altogether, we obtain $\boxed{14}$ values of $n \leq 100$ for which $\tau(n) + \tau(2n) = 18$.