Contest Solutions (Europe/Africa)

Middle School Division

Saturday, March 26, 2022

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1. Assuming that 1 meter is equal to 100 centimeters, 1 yard is equal to 3 feet, 1 foot is equal to 12 inches, and 40 inches is equal to 1 meter, how many centimeters are equal to one yard?

Answer: 90

One yard is equal to 3 feet, which is equal to 12 \cdot 3 = 36 inches. 36 inches is \(\frac{9}{10}\) of 40 inches, so it is \(\frac{9}{10}\) or 90 percent, of one meter, meaning that it is equal to 90 centimeters.

2. Blanche writes, for each \(1 \leq k \leq 26\), \(k\) copies of the \(k^{th}\) letter of the alphabet in a row, so that her string begins ABBCCDDEEEE; and ends with 26 Z’s. What is the middle letter in her string?

(a) M
(b) P
(c) R
(d) T
(e) none of the above

Answer: E

There are a total of \(1 + 2 + 3 + \cdots + 26 = 351\) letters in Blanche’s string; we want the 176th. That is, we want the smallest positive integer \(n\) such that \(1 + 2 + 3 + \cdots + n \geq 176\). Since \(1 + 2 + 3 + \cdots + 18 = 171\) and \(1 + 2 + 3 + \cdots + 19 = 190\), we know that \(n = 19\). The 19th letter of the alphabet is S.

3. The circumference of a 150° sector of a circle with integer radius is \(15\pi + k\) for some integer \(k\). What is \(k\)?

(a) 18
(b) 24
(c) 36
(d) 60
(e) none of the above

Answer: C

As a 150° sector of a circle has arc length \(\frac{150}{360} = \frac{5}{12}\) times that of the entire circle, \(15\pi\) is \(\frac{5\pi}{12}\) the circumference of the entire circle, or \(36\pi\). As the circumference of a circle with radius \(r\) is \(2\pi r\), this implies that \(r = 18\); because the sector consists of an arc length and two radii, we have that \(k = 2r = 36\).

4. The sum of the square roots of three distinct positive integers \(a\), \(b\), and \(c\) summing to 35 is an integer. Compute the product \(abc\).

Answer: 225

All of \(a\), \(b\), and \(c\) must be perfect squares, so they are each one of 1, 4, 9, 16, or 25. In order for \(a\), \(b\), and \(c\) to sum to 35, they must be (some permutation of) 1, 9, and 25, which can be seen by inspection. Hence, \(abc = 225\).

5. How many positive integers between 1 and 100, inclusive, are the positive difference between two numbers of the form \(N^2 - N + 1\) for some positive integer \(N\)?

Answer: 50

All even positive integers can be expressed as such a difference; consider \((n + 1)^2 - (n + 1) + 1) - (n^2 - n + 1) = 2n\). Note, however, that \(N^2 - N\) is always even, so \(N^2 - N + 1\) is always odd. An
odd number cannot be the positive difference between two odd numbers, so we have found all desired integers. There are 50 even integers between 1 and 100, inclusive.

6. Sixty times a positive integer leaves a remainder of 58 when divided by 119. Compute the smallest possible value of this positive integer.

Answer: 116

Call the positive integer \( n \); then if \( 60n \equiv 58 \pmod{119} \), \( 120n \equiv 116 \pmod{119} \), or \( n \equiv 116 \pmod{119} \). This makes the smallest possible value of \( n \) equal to 116.

7. Triangle \( ABC \) has \( AB = AC = 10 \) and \( BC = 12 \). Point \( D \) lies on \( BC \) between \( B \) and \( C \) with \( BD = 10 \). Compute \( AD^2 \).

Answer: 80

By dropping the altitude from \( A \) to the midpoint \( M \) of \( BC \), we observe that \( AM = 8 \) and \( BM = MC = 6 \). Thus, \( MD = 10 - 6 = 4 \), and by the Pythagorean theorem, \( AD^2 = 8^2 + 4^2 = 80 \).

8. Chester flips 4 fair coins, and Rhiannon flips 6 fair coins. What is the probability that Rhiannon flips more heads than Chester? Express your answer as a common fraction.

Answer: \( \frac{319}{512} \)

It’s well-known (by symmetry) that the probability of Rhiannon flipping more heads than Chester if Chester flips \( c \) coins and Rhiannon flips \( c + 1 \) coins is \( \frac{1}{2} \). (The sketch of the argument uses the symmetry of the choose function: if 0 of Chester’s flips are heads, all Rhiannon needs to do is flip one head, i.e. not flip all tails; but if all of Chester’s flips are heads, Rhiannon must flip all heads. Likewise, in general, if Chester flips \( h < c \) heads, Rhiannon must flip at least \( h + 1 \) heads; but if Chester were to flip \( c - h \) heads, Rhiannon would need to not flip at least \( c - h \) tails.) Given that this event with probability \( \frac{1}{2} \) occurs (with \( c = 4 \), the sixth coin is irrelevant: the problem condition is already satisfied. Otherwise, Rhiannon’s sixth coin can only push her over Chester’s total if Chester and Rhiannon are tied with Chester having flipped 4 coins and Rhiannon having flipped 5 coins. The probability this occurs is \( \frac{1}{2} \) of the sum of probabilities \( \frac{\binom{4}{0}\binom{5}{1} + \binom{4}{1}\binom{5}{2} + \binom{4}{2}\binom{5}{3} + \binom{4}{3}\binom{5}{4}}{2^9} = \frac{1+20+60+40+5}{512} = \frac{63}{256} \), hence \( \frac{63}{512} \). Adding this to the probability of \( \frac{1}{2} \) that Rhiannon has more heads than Chester after she flips 5 coins and Chester flips 4 gives our final probability of \( \frac{319}{512} \).

9. Square \( MATH \) has side length 2. Point \( P \) lies on \( MA \) such that the area of quadrilateral \( PATH \) is 3.9. Compute the area of overlap between quadrilateral \( PATH \) and triangle \( MTH \). Express your answer as a common fraction.

Answer: \( \frac{40}{21} \)

We can draw the following diagram:
Call $Q$ the intersection point of $MT$ and $PH$. Since we know that $\frac{MP}{MA} = \frac{1}{20}$, we have $\frac{MQ}{QT} = \frac{1}{20}$ by similarity, whence $QT = \frac{20}{21} MT$, and the area of $\triangle QTH$ is $\frac{20}{21}$ times that of $\triangle MTH$, which is just 2.

Thus, the desired area is $\frac{40}{21}$.

10. Compute the sum of all integers $n$ such that

$$\frac{\sqrt{n} + 5}{\sqrt{n} + 1}$$

is an integer.

Answer: 10

The given quantity can be rewritten as $1 + \frac{4}{\sqrt{n} + 1}$, so we want $\frac{4}{\sqrt{n} + 1}$ to be an integer. As $\sqrt{n} + 1$ must be greater than 1, this requires that $\sqrt{n} + 1 = 1, 2, 4$, so that $n = 0, 1, 9$ respectively, which sum to 10.

11. An isosceles triangle has a base length of 10 and two side lengths of 13. A point inside the triangle is equidistant with common distance $d$ from all three vertices of the triangle. Then $d$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

Answer: 193

Call the triangle $ABC$, with $AB = AC = 13$ and $BC = 10$. Let $P$ be the point inside $\triangle ABC$ with $PA = PB = PC$. Then $P$ must be at a horizontal distance of 5 from both $B$ and $C$; suppose its distance from $BC$ is $h$. Then $d^2 = 25 + h^2$, and is also equal to $(12 - h)^2$, since we can split $\triangle ABC$ down its $A$-altitude into two 5-12-13 right triangles (where the altitude length is 12). Thus, $25 + h^2 = 144 - 24h + h^2$, from which we get $h = \frac{119}{24}$ and the common distance $d$ as $12 - h = 12 - \frac{119}{24} = \frac{169}{24}$, so $m + n = 169 + 24 = 193$.

12. A positive integer is called stable if none of its digits are greater than the cube of the smallest digit. Compute the number of stable positive integers less than or equal to 1000.

Answer: 543

Say the smallest digit is 0; then all digits must be 0, which is a contradiction. If the smallest digit is 1, all digits must be 1, which gives 1, 11, and 111 as stable numbers. If the smallest digit is 2, all digits must lie between 2 and 8, inclusive, so we have 2, 22 through 28, and the numbers from 32 through 82 ending in 2 as two-digit stable numbers, of which there are 14.

For three-digit stable numbers with a smallest digit of 2, if 2 is the hundreds digit, we have $7^2 = 49$ stable numbers: namely, numbers of the form $2xy$ where $2 \leq x, y \leq 8$. If 2 is the tens digit, we have 323 through 328, 423 through 428, 523 through 528, 623 through 628, 723 through 728, and 823 through 828, for another $6^2 = 36$ stable numbers. If 2 is the units digit, we similarly get another $6^2 = 36$ stable numbers. We also have 6 numbers of the form $x22$ (for $3 \leq x \leq 8$) which have not yet been counted.

Finally, if the smallest digit is at least 3, there are no other restrictions on the digits, so 3 through 9, the $7^2 = 49$ two-digit positive integers with digits from 3-9, and the $7^3 = 343$ three-digit positive integers with digits from 3-9. Altogether, we have $3 + 14 + 49 + 36 + 36 + 6 + 7 + 49 + 343 = 543$ stable numbers less than or equal to 1000.

13. Given that

$$7 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{673} < 8,$$
compute the largest integer not exceeding
\[
\frac{3}{1+2} + \frac{6}{4+5} + \frac{9}{7+8} + \frac{12}{10+11} + \cdots + \frac{2022}{2020+2021}.
\]
Answer: 339

This is equal to
\[
1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \cdots + \frac{674}{1347} = 1 + \left( \frac{1}{2} + \frac{1}{6} \right) + \left( \frac{1}{2} + \frac{1}{10} \right) + \left( \frac{1}{2} + \frac{1}{14} \right) + \cdots + \left( \frac{1}{2} + \frac{1}{2694} \right).
\]
Noting that there are 674 terms in the sum, all of the \(\frac{1}{2}\)'s sum to \(\frac{1}{2} \cdot 674 = 337\). Along with the 1, we have a sum of 338 up to this point.

The remaining terms sum to
\[
\frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \cdots + \frac{1}{2694} = \frac{1}{1 \cdot 4 + 2} + \frac{1}{2 \cdot 4 + 2} + \frac{1}{3 \cdot 4 + 2} + \cdots + \frac{1}{673 \cdot 4 + 2}.
\]
We can bound this sum \(S\) strictly between
\[
\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{673 \cdot 4}
\]
and
\[
\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 4} + \cdots + \frac{1}{674 \cdot 4}.
\]
The first sum is \(\frac{1}{4}\) of \(H_{673}\), where \(H_n\) is defined as the sum \(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\), and the second sum is \(\frac{1}{4}\) of the quantity \(H_{673} - 1\). By the problem statement, we know that \(7 < H_{673} < 8\), so \(\frac{7}{4} < \frac{1}{4}H_{673} < 2\). Similarly, \(6 + \frac{1}{674} < H_{674} - 1 < 7 + \frac{1}{674}\), so \(\frac{7}{4} + \frac{1}{674} < \frac{1}{4}(H_{674} - 1) < \frac{7}{4} + \frac{1}{674} < 2\). In other words, we know that \(1 < S < 2\), so 338 + \(S\) is between 339 and 340, and the largest integer not exceeding the desired sum is 339.

In general, we can observe that
\[
H_n < 1 + \left( \frac{1}{2} + \frac{1}{2} \right) + \left( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8} \right) + \cdots
\]
(with eight \(\frac{1}{8}\)'s), with each block summing to 1, which shows that \(H_{2^n-1} < n\). Since \(2^9 - 1 < 673 < 2^{10} - 1\), we know that \(H_{673} < 10\). At the same time,
\[
H_n > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \cdots,
\]
meaning that \(H_{2^n} > 2 + \frac{n}{2}\) also.

14. Let rectangle \(ABCD\) have \(AB = 3\) and \(BC = 6\). Points \(E, F, G,\) and \(H\) lie on \(AB, BC, CD,\) and \(DA\) respectively such that \(EB = GD = 1\) and \(EF = GH\). If the perimeter of \(AEFCHG\) is at most 90 percent of the perimeter of \(ABCD\), compute the maximum possible length of \(FC\).

Answer: \(\frac{21}{20}\)

We draw the following diagram.
15. The digital root of a positive integer is the result of repeatedly summing the digits of that integer until a single integer from 1 to 9, inclusive, is obtained. For example, the digital root of 2022 is 6, the digital root of 1234567 is 1 (since the sum of digits is 28, 2 + 8 = 10, and 1 + 0 = 1), and the digital root of 36 is 9. Compute the sum of the digital roots of all the positive integers from 1 to 2022, inclusive.

Answer: 10101

We claim the digital root of \( n \) is congruent to \( n \mod 9 \) and is an integer in \([1, 9]\). It suffices to show that the digit sum is congruent to \( n \mod 9 \), since the digital root is obtained from repeatedly iterating the digit sum operation. Recall that any base-10 positive integer \( n \) can be written as a sum of powers of 10 multiplied with each of the corresponding digits, and since \( 10^k \equiv 1 \mod 9 \) for all non-negative integers \( k \), the sum of digits is congruent to the sum of \( 10^k \) times the digits over all \( 0 \leq k \leq m \), where \( m \) is the number of digits of \( n \).

To compute the sum of the digital roots of the integers from 1 to 2022, note that we can sum the remainders when each of them is divided by 9, except for the multiples of 9, which have digital roots of 9 rather than 0. The sum of the digital roots in each block of nine is \( 1 + 2 + 3 + \cdots + 9 = 45 \). The largest multiple of nine less than or equal to 2022 is 1998 = 9 \cdot 222, so we have a sum of digital roots of 1998 up to, and including, 2022. Finally, the sum of the digital roots of 2017 through 2022 is \( 1 + 2 + 3 + 4 + 5 + 6 = 21 \), so we get a final sum of 1998 + 21 = 10101.

16. A set consists of sixteen distinct positive integers which sum to 139. When one of these sets is chosen uniformly at random, compute the expected value of its largest element.

Answer: 18

Observe that \( 1 + 2 + 3 + \cdots + 16 = 136 \) and that \( 1 + 2 + 3 + \cdots + 17 = 153 \), so the set may be one of \{1, 2, 3, \cdots, 15, 19\}, \{1, 2, 3, \cdots, 14, 16, 18\}, or \{1, 2, 3, \cdots, 13, 15, 16, 17\}. If we tried setting the maximal element to 20 or larger, the smallest 15 elements, which would need to sum to at least \( 1 + 2 + 3 + \cdots + 15 = 120 \), would force the sum of the elements in such a set to be at least 140, which is a contradiction. The expected value of the largest element is then \( \frac{19 + 18 + 17}{3} = 18 \).
17. For each positive integer $n$, define

$$f(n) = \frac{\sum_{i=1}^{n} (i + i^2)}{\sum_{i=1}^{n} i^3}.$$ 

Compute the smallest positive integer $n$ for which $f(n) \leq \frac{1}{10}$.

Answer: $15$

The numerator simplifies to

$$\left( \sum_{i=1}^{n} i \right) + \left( \sum_{i=1}^{n} i^2 \right) = \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{3},$$

while the denominator is $\left( \frac{n(n+1)}{2} \right)^2$. Thus, $f(n)$ simplifies to $\frac{4}{3} \cdot \frac{n+2}{n^2+n}$. In order for this to be at most $\frac{1}{10}$, we must have $\frac{n+2}{n^2+n} \leq \frac{3}{10}$, or $40n + 80 \leq 3n^2 + 3n \implies 3n^2 - 37n - 80 \geq 0$. This occurs when $n \geq 15$ (by the quadratic formula).

18. Let $ABCD$ be a rectangle with $AB = 2$ and $BC = 1$. Suppose that $E$ and $F$ are points on $\overline{AB}$ and $\overline{CD}$, respectively, lying on the same side of $\overline{BC}$, such that $AE \cdot CF = 1$ and $EF = \frac{5}{4}$. The largest possible length of $CF$ can be written in the form $\frac{p+\sqrt{q}}{r}$, where $p$, $q$, and $r$ are positive integers with $q$ not divisible by the square of a prime. Compute $p + q + r$.

Answer: $204$

By the Pythagorean theorem, the horizontal distance between $E$ and $F$ is $\sqrt{\left( \frac{5}{4} \right)^2 - 1^2} = \frac{3}{4}$. Letting $AE = x$ (with $BE = x + 2$) and $CF = x + \frac{11}{4}$, we solve the equation $x(x + \frac{11}{4}) = 1$, and get that $4x^2 + 11x - 4 = 0$, or $x = \frac{-11 \pm \sqrt{177}}{8}$. Taking the positive value, we get that $p + q + r = 11 + 185 + 8 = 204$.

19. Let $n$ be a positive integer. A permutation of $(a_1, a_2, a_3, \dots, a_{2n})$ is called rightweight if $2(a_1 + a_2 + a_3 + \cdots + a_n) \leq a_{n+1} + a_{n+2} + a_{n+3} + \cdots + a_{2n}$. Compute the number of permutations of $(1, 2, 3, 4, 5, 6, 7, 8)$ that are rightweight.

Answer: $2304$

The sum of all elements is 36, so we require $a_1 + a_2 + a_3 + a_4 \leq 12$. The possible 4-tuples $(a_1, a_2, a_3, a_4)$ are the permutations of $(1, 2, 3, 4)$, $(1, 2, 3, 5)$, $(1, 2, 3, 6)$, and $(1, 2, 4, 5)$, giving $4!^2 \cdot 4 = 2304$ rightweight permutations.

20. Let $\tau(n)$ denote the number of positive integer divisors of $n$. Compute the number of positive integers $n \leq 100$ satisfying $\tau(n) + \tau(2n) = 18$.

Answer: $14$

In the first case, $n$ is odd, in which case $\tau(2n) = 2\tau(n)$ and $\tau(n) = 6$. Noting that the number of divisors of a positive integer is equal to the products of the terms of the form $1 + k_i$, where prime factor $p_i$ has exponent $k_i$, we observe that $n$ can be of the forms $p^5$ or $p^2q$ for primes $p \neq q$. In the first case, we have no values of $n$ (since $n = 2^5 = 32$ is even, and for $p \geq 3$, we get $n > 100$); in the latter, we may have $n = 3^2 \cdot 5 = 45$, $3^2 \cdot 7 = 63$, $3^2 \cdot 11 = 99$, or $5^2 \cdot 3 = 75$.

In the second case, $n$ has a power of $2^k$ in its prime factorization for some integer $k \geq 1$, implying that $\tau(2n) = \frac{2k+1}{2} \tau(n)$. This means that $\tau(n) \left( \frac{4k+4}{2k+1} \right) = 18$, so $2k + 3$ must divide $18(k + 1) = 18k + 18$. By the Euclidean algorithm, $2k + 3 \mid (18k + 18) \iff (2k + 3) \mid -9$, so $k = 3$ is the only possible value, implying that $\tau(n) = 8$. Then $n = 2^7$, $2^3 \cdot p$ for some prime $p$, $2p^3$, or $2pq$ for distinct primes $p$ and $q$. 
As $2^7 > 100$, we discard this subcase. For $n = 2^3 \cdot p$, we have $n \in \{16, 24, 40, 56, 88\}$, and for $n = 2pq$, we have $n \in \{30, 42, 66, 78, 70\}$. This gives 10 additional values of $n$. Hence, altogether, we obtain 14 values of $n \leq 100$ for which $\tau(n) + \tau(2n) = 18$. 