Online Open Tournament

Contest Solutions (North/Central/South America)

Middle School Division

Saturday, March 26, 2022

The problems and solutions for this competition were prepared by the teaching staff of CyberMath Academy.
Thanks to Thinula de Silva and Freya Edholm for reviewing these problems and solutions!

1. If $x = |x + 20| - 22$, compute x.

   Answer: $\boxed{-21}$

   We have $x + 22 = |x + 20|$; note that $x + 20$ must be negative, since $x + 22 \neq x + 20$. This means that $|x + 20| = -(x + 20) = -x - 20$, and solving $x + 22 = -x - 20$, we get $2x = -42$, or $x = \boxed{-21}$.

2. Compute the number of positive integers k for which $5 < \frac{k}{1000-k} < 6$.

   (a) 20
   (b) 22
   (c) 23
   (d) 25
   (e) none of the above

   Answer: $\boxed{E}$

   We have from $5 < \frac{k}{1000-k}$ that $5(1000 - k) = 5000 - 5k < k$, or $5000 < 6k \implies k \geq 834$. On the other hand, we also have $k < 6(1000 - k) = 6000 - 6k$, so that $7k < 6000$, or $k \leq 857$. This gives $857 - 834 + 1 = \boxed{24}$ possible positive integer values of k.

3. For a two-digit positive integer N, Byron subtracts from N the product of its digits. The result is 74. What was Byron's original choice for the integer N?

   Answer: $\boxed{92}$

   Let $N = \overline{ab} = 10a + b$ for digits $7 \leq a \leq 9$ and $0 \leq b \leq 9$. If $a = 7$, then ab must be a multiple of 7, but this forces $N \geq 81$, which is a contradiction. If $a = 8$, we have 82 as a candidate, but the product of digits is 16, not 8. Finally, if $a = 9$, we find that $b = 2$ works, so $N = \boxed{92}$.

4. It is currently 2:00. After how many minutes will the hour and minute hands on a 12-hour clock form a $45°$ angle between them for the first time? Express your answer as a common fraction.

   Answer: $\boxed{\dfrac{30}{11}}$

   The angle between the hands at 2:00 is $60°$ (or, more precisely, the hour hand is at an angle of $+60°$ relative to the minute hand). With every minute that passes, the hour hand moves forward by $\frac{1}{60}$ of one hour, or $30°$ along the clock face, which is $0.5°$, and the minute hand moves $\frac{1}{60}$ of the way around the clock, or $6°$. Therefore, they grow $5.5°$ closer (or further apart) every minute. For the hands to get $15°$ closer, it would take $\frac{15}{5.5} = \boxed{\dfrac{30}{11}}$ minutes.

5. A video on the video sharing site CyberCast has 100 more likes than dislikes. The *score* of a video is computed as $1.2l - 0.9d$, where l and d are the numbers of likes and dislikes, respectively. If the score of the video is 285, and each vote is either a like or a dislike, compute the total number of votes on the video.

   (a) 600
   (b) 900
   (c) 1050
   (d) 1200
   (e) none of the above

Answer: $\boxed{D}$

We have $l = d + 100$, so the score in terms of the number of dislikes d is $1.2(d+100) - 0.9d = 0.3d + 120$. Setting this equal to 285 gives $d = 550$, and thus, $l = 650$, so the total number of votes is $l + d = \boxed{1200}$.

6. The sum of the squares of the first n positive integers ends in 6. Compute the smallest possible value of n.

Answer: $\boxed{11}$

We have $\frac{n(n+1)(2n+1)}{6} \equiv 6 \bmod 10$. This implies that $n(n+1)(2n+1) \equiv 6 \bmod 10$, or to $0 \bmod 2$ and $1 \bmod 5$ by the Chinese remainder theorem. We note that $n(n+1)(2n+1)$ is always even, so it suffices to find the smallest n for which $n(n+1)(2n+1) \equiv 1 \bmod 5$. We can't have $n \equiv 0, 2, 4 \bmod 5$, for these would make n, $2n+1$, and $n+1$ multiples of 5, respectively. Additionally, for $n \equiv 3 \bmod 5$, we get that $n(n+1)(2n+1) \equiv 3 \cdot 4 \cdot 7 \equiv 4 \bmod 5$, so we rule this out as well. Trying $n = 1, 6, 11$, we find that $n = \boxed{11}$ is the smallest possible value.

7. What is the average value of the base-9 positive integers less than 100_9, when they are read as base-10 integers?

Answer: $\boxed{\dfrac{891}{20}}$

This is

$$\frac{1+2+3+\cdots+8+10+11+12+\cdots+18+\cdots+80+81+82+\cdots+88}{80}$$
$$= \frac{36+9(14+24+34+\cdots+84)}{80} = \frac{36+9\cdot(98\cdot 4)}{80} = \frac{3564}{80} = \boxed{\frac{891}{20}}.$$

8. The letters in CYBERMATH are each written down on their own slip of paper and tossed into a bag. Three slips of paper are then drawn out of the bag, without replacement. Compute the probability that at least two vowels (A, E, I, O, U, or Y) are drawn. Express your answer as a common fraction.

Answer: $\boxed{\dfrac{19}{84}}$

There are 3 vowels, so the probability that three vowels are drawn is $\frac{1}{\binom{9}{3}} = \frac{1}{84}$ (since all letters are distinct). The probability that two vowels are drawn is $\frac{\binom{3}{2}\binom{6}{1}}{\binom{9}{3}} = \frac{18}{84}$, the probability that one vowel is drawn is $\frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} = \frac{45}{84}$, and the probability that no vowels are drawn is $\frac{\binom{6}{3}}{\binom{9}{3}} = \frac{20}{84}$. The probability of drawing at least two vowels is therefore $\boxed{\dfrac{19}{84}}$.

9. For every positive integer n, let $Q(n)$ be the closest integer to $\sqrt{n}$. What is the value of $Q(1) + Q(2) + Q(3) + \cdots + Q(100)$?

Answer: $\boxed{670}$

We have $1.5^2 = 2.25$, $2.5^2 = 6.25$, $3.5^2 = 12.25$, and so forth, with $8.5^2 = 72.25$ and $9.5^2 = 90.25$, so that $Q(1) = Q(2) = 1$, $Q(3) = Q(4) = Q(5) = Q(6) = 2$, $Q(7) = Q(8) = \cdots = Q(12) = 3$, and so on until $Q(73) = Q(74) = Q(75) = \cdots = Q(90) = 9$ and $Q(91) = Q(92) = Q(93) = \cdots = Q(100) = 10$. Altogether, we get a sum of $1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 + \cdots + 9 \cdot 18 + 10 \cdot 10 = \boxed{670}$.

10. If x and y are positive real numbers so that $x^2 + y^2 = 74$ and $x + y = 12$, compute $x^3 + y^3$.

Answer: $\boxed{468}$

Since $x^2+y^2 = (x+y)^2 - 2xy = 144 - 2xy$, we get $xy = 35$. Then $x^3+y^3 = (x+y)^3 - (3x^2y+3xy^2) = 1728 - 3xy(x+y) = 1728 - 3 \cdot 35 \cdot 12 = \boxed{468}$.

11. Suppose f is a quadratic polynomial with $f(0) = 1$, $f(1) = 20$, and $f(2) + f(3) = 22$. Then the coefficient of its linear term can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: $\boxed{235}$

Let $f(x) = ax^2 + bx + c$ for constants a, b, and c with $a \neq 0$. Plugging in $x = 0$, we get $f(0) = c = 1$. Plugging in $x = 1$, we get $a+b+c = 20$, so $a+b = 19$. Plugging in $x = 2$ and $x = 3$, we get $(4a+2b+c)+(9a+3b+c) = 22$, so $13a+5b+2c = 22$ and $13a+5b = 20$. Solving the system $a+b = 19$, $13a+5b = 20$ gives $(a,b) = \left(-\frac{75}{8}, \frac{227}{8}\right)$. Here, $b = \frac{237}{8}$ is the coefficient of the linear term, and $m+n = 227+8 = \boxed{235}$.

12. Let m and n be integers for which n is a multiple of m and $\frac{mn}{m+n} = \frac{10}{3}$. Compute the sum of the possible values of mn.

Answer: $\boxed{40}$

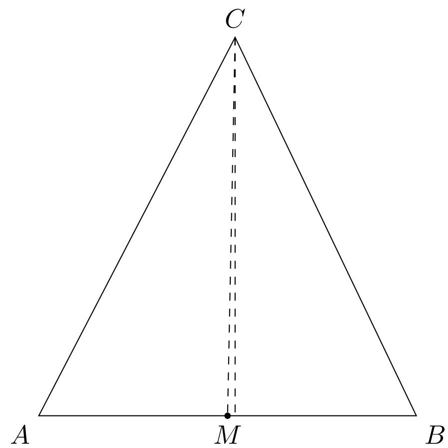
We have $3mn = 10m+10n$, or $3mn - 10m - 10n = 0 \implies 9mn - 30m - 30n = 0$. By Simon's favorite factoring trick, we can write this as $(3m-10)(3n-10) = 100$. Since $3m-10$ and $3n-10$ are both integers that are congruent to 2 mod 3, we want them to be integer factors of 100 congruent to 2 mod 3 (up to sign). These are $-1, 2, 5, 20$, and $3m - 10 = 2, 5, 20, -2, -5, -20$ yield $m = 4, 5, 10, \frac{8}{3}, \frac{5}{3}, -\frac{10}{3}$, respectively. The integer values of m, $m = 4$, $m = 5$, and $m = 10$, yield $n = 20$, $n = 10$, and $n = 5$, respectively. As 20 is a multiple of 4 and 10 is a multiple of 5, but 5 is not a multiple of 10, and 30 is a multiple of -3, the resulting distinct values of mn are 80, 50, and -90, which sum to $\boxed{40}$.

Another method is to set $n = km$ for some integer k, from which $\frac{mn}{m+n} = \frac{km^2}{m(1+k)} = \frac{10}{3}$, and $\frac{k}{1+k} \cdot m = \frac{10}{3}$. Since $\frac{1}{2} \leq \frac{k}{1+k} < 1$ if $k > 0$, and otherwise $1 < \frac{k}{1+k} \leq 2$ if $k < -1$, we obtain $m \in \{2, 3, 4, 5, 6\}$. Of these, only $m = 3$, $m = 4$, and $m = 5$ work, and these give $n = -30$, $n = 20$, and $n = 10$ as before, so the sum of the possible values of mn is again $\boxed{40}$.

13. Line segment $\overline{AB}$ has side length 5, and point C lies in the plane of $\overline{AB}$ with $AC^2 = CB^2 + 1$. Point M is the midpoint of $\overline{AB}$. If $MC = 5$, the square of the distance from C to $\overline{AB}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: $\boxed{2599}$

We can draw the following diagram:



With D the foot of the perpendicular from C to $\overline{AB}$ as above, $CD = h$, $DA = x$, and $DB = 5 - x$, we obtain $x^2 + h^2 = (5-x)^2 + 1$, so $x^2 + h^2 = 26 - 10x + x^2 + h^2$, and $x = \frac{13}{5}$. So $MD = \frac{1}{10}$, from which we get $DC^2 = \frac{2499}{100}$ and $m + n = 2499 + 100 = \boxed{2599}$.

14. Let a and b be real numbers for which $\frac{a^2+b^2}{ab} = 7$ and $a + b = 6$. Compute $\max(a, b)$. Express your answer in simplest radical form.

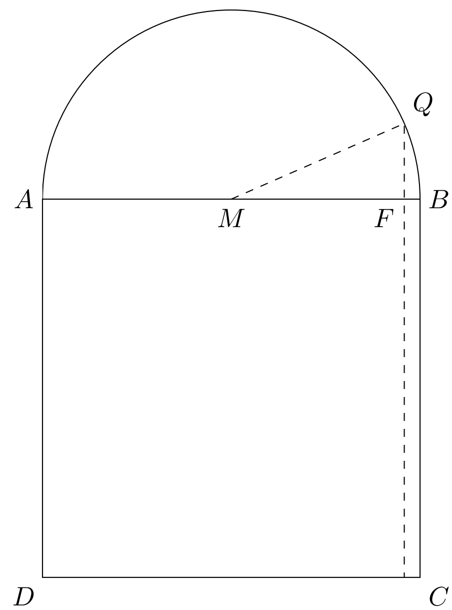
Answer: $\boxed{3 + \sqrt{5}}$

From $a^2 + b^2 = 7ab$, we can write $a^2 + 2ab + b^2 = (a+b)^2 = 9ab$, and thus $ab = 4$. Solving the resulting system $a + b = 6$, $ab = 4$ yields $a(6 - a) = 4$, or $a^2 - 6a + 4 = 0 \implies a = 3 \pm \sqrt{5}$, with $b = 3 \mp \sqrt{5}$. Among these, $\boxed{3 + \sqrt{5}}$ is the larger value.

15. Square $ABCD$ has side length 1. A semicircle has arc AB and center at the midpoint of $\overline{AB}$. Point Q lies on the circumference of the semicircle so that the distance from Q to $\overline{CD}$ is 1.2. If Q is closer to C than to D, the value of QC^2 can be written in the form $\frac{m-\sqrt{n}}{p}$, where m, n, and p are positive integers, and n is not a multiple of the square of any prime. Compute $m + n + p$.

Answer: $\boxed{50}$

We have the following diagram:



Labeling the midpoint of $\overline{AD}$ as M, and noting that $MQ = \frac{1}{2}$, we have $MF = \frac{\sqrt{21}}{10}$, and thus, $FB = \frac{5-\sqrt{21}}{10}$. Then $QC^2 = \left(\frac{6}{5}\right)^2 + FB^2 = \frac{19-\sqrt{21}}{10}$, from which we obtain $p+q+r = 19+21+10 = \boxed{50}$.

16. A magical tetrahedral die initially has 1 red face and 3 blue faces, each equally likely to be rolled. Whenever a red face comes up, one of the blue faces changes to a red face. Compute the expected number of rolls before all faces of the die turn red.

Answer: $\boxed{\dfrac{22}{3}}$

The expected number of rolls before rolling the first red face is $\frac{1}{\frac{1}{4}} = 4$, by the linearity of expectation (in general, an event with probability p of occuring is such that the expected number of the trial on which it occurs is $\frac{1}{p}$). The expected number of rolls of the new 2-red, 2-blue die before a red face comes up is 2, and the expected number of rolls of the resulting 3-red, 1-blue die before a red face comes up is $\frac{4}{3}$. Altogether, by linearity of expectation, this gives an expected number of rolls of $4 + 2 + \frac{4}{3} = \boxed{\dfrac{22}{3}}$.

17. What is the area of a triangle whose side lengths are $\sqrt{2}$, 3, and $\sqrt{11 + 2\sqrt{3} + 2\sqrt{6}}$? Express your answer as a common fraction in simplest radical form.

Answer: $\boxed{\dfrac{\sqrt{6}-\sqrt{3}}{2}}$

Without loss of generality, let the coordinates of the vertices be $(0, 0)$, (a, b), and (c, d), with $a^2+b^2 = 2$, $(c - a)^2 + (d - b)^2 = 9$, and $c^2 + d^2 = 11 + 2\sqrt{3} + 2\sqrt{6}$. Combining these equations, we get $ac + bd = 2 + \sqrt{3} + \sqrt{6}$. One might reasonably guess that $(a, b, c, d) = (1, 1, 1 + \sqrt{3}, 1 + \sqrt{6})$. By Shoelace, the triangle's area is $\frac{1}{2} |ad - bc|$, which evaluates to $\boxed{\dfrac{\sqrt{6}-\sqrt{3}}{2}}$.

18. The sum of two positive integer perfect squares ends in the digits 34. Compute the second-smallest possible sum of the square roots of the integers.

Answer: $\boxed{18}$

We know that $(3, 5)$ gives the smallest possible sum of 8; we can also check that $(225, 9)$ works, so the answer is no more than 18. No pair of perfect squares sums to 134, and if x^2 and y^2 with $x^2 + y^2 = 234$ were closer together, the sum $x + y$ would exceed 18. So $\boxed{18}$ is the second-smallest possible value of $x + y$.

19. Suppose that

$$\sum_{n=2}^{10} \frac{n^2}{\binom{n}{2}\binom{n+1}{2}} = \frac{p}{q}$$

for relatively prime positive integers p and q. Compute $p + q$.

Answer: $\boxed{199}$

The sum telescopes as

$$\sum_{n=2}^{10} \left(\frac{1}{\binom{n}{2}} + \frac{1}{\binom{n+1}{2}} \right),$$

as we can see by obtaining a common denominator:

$$\frac{1}{\binom{n}{2}} + \frac{1}{\binom{n+1}{2}} = \frac{\binom{n+1}{2} + \binom{n}{2}}{\binom{n+1}{2}\binom{n}{2}} = \frac{\frac{(n+1)n+n(n-1)}{2}}{\binom{n+1}{2}\binom{n}{2}} = \frac{n^2}{\binom{n+1}{2}\binom{n}{2}}.$$

Notice that our sum is

$$2\left(\frac{1}{\binom{2}{2}} + \frac{1}{\binom{3}{2}} + \frac{1}{\binom{4}{2}} + \cdots + \frac{1}{\binom{11}{2}} \right) - \left(\frac{1}{\binom{2}{2}} + \frac{1}{\binom{11}{2}} \right) = 2\left(\frac{1}{\binom{2}{2}} + \frac{1}{\binom{3}{2}} + \frac{1}{\binom{4}{2}} + \cdots + \frac{1}{\binom{11}{2}} \right) - \frac{56}{55}.$$

To finish, we observe that

$$\sum_{k=2}^{n} \frac{1}{\binom{k}{2}} = 2 - \frac{2}{n},$$

which can be proven by induction. Then our sum is just $2\left(2 - \frac{2}{11}\right) - \frac{56}{55} = \frac{144}{55}$, from which $p + q = 144 + 55 = \boxed{199}$.

20. A set S of positive integers is said to be *relatively open* if the set $T := \{s_1 + s_2 : s_1, s_2 \in S, s_1 \neq s_2\}$ is missing at least half of the positive integers from $\min(S)$ to $\max(S)$, inclusive. For example, $S = \{1, 2, 3, 10\}$ is relatively open, because $T = \{3, 4, 5, 11, 12, 13\}$ is missing 1 and 6-10 among $[1, 10]$, whereas $S = \{1, 2, 3, 5, 6, 10\}$ would not be relatively open, because $T = \{3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16\}$ is only missing 1, 2, and 10 among $[1, 10]$. Compute the number of nonempty subsets of $\{1, 2, 3, 4, 5, 6\}$ that are relatively open.

Answer: $\boxed{58}$

Note that the sum of any two distinct elements must be at least 3. If $\min(S) = 1$ and $\max(S) = 6$ (both 1 and 6 are in S), the condition of S being relatively open is equivalent to its sumset T not having all of 3-6 as elements. If two or fewer of $\{2, 3, 4, 5\}$ are in S, there are only $\binom{3}{2} = 3$ sums of elements in S that do not include 6, so such sets are necessarily relatively open. If all four elements from $\{2, 3, 4, 5\}$ were to belong to S, $T = \{3, 4, 5, 6, 7, 8, 9\}$. If $5 \notin S$ or $4 \notin S$, then T contains $\{3, 4, 5, 6\}$. If $3 \notin S$ or $2 \notin S$, then S will be relatively open. This gives 13 relatively open sets in this case.

If $\min(S) = 1$ and $\max(S) = 5$, S will be relatively open if it is $\{1, 5\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{1, 2, 4, 5\}$, or $\{1, 3, 4, 5\}$, adding 6 to our total of relatively open sets S. If $\min(S) = 2$ and $\max(S) = 6$, the smallest possible element of T is $2 + 3 = 5$, so all 8 possible sets S in this case are relatively open.

If $\max(S) - \min(S) \leq 3$, then S is relatively open, because for $\max(S) - \min(S) = 3$, with, say,

$\min(S) = m$ and $\max(S) = m + 3$, there are two numbers between $\max(S)$ and $\min(S)$, namely $m + 1$ and $m + 2$, but $2m + 1$ can be at most 1 less than $m + 3$, meaning that T contains at most 2 elements. But $|S| = 4$, meaning that S is relatively open. Similar reasoning applies to $\max(S) - \min(S) = 2, 1, 0$, for yet another $3 \cdot 4 + 4 \cdot 2 + 5 \cdot 1 + 6 = 31$ relatively open sets S.

Altogether, there are $13 + 6 + 8 + 31 = \boxed{58}$ relatively open subsets of $\{1, 2, 3, 4, 5, 6\}$.

TB. How many 10-digit positive integers have digits that sum to exactly 10?

Answer: $\boxed{48619}$