cyber math

Online Open Tournament

Contest File (North/Central/South America)

Middle School Division

Saturday, March 26, 2022

1. If $x = |x + 20| - 22$, compute x.
2. Compute the number of positive integers k for which $5 < \frac{k}{1000-k} < 6$.
   (a) 20
   (b) 22
   (c) 23
   (d) 25
   (e) none of the above
3. For a two-digit positive integer N, Byron subtracts from N the product of its digits. The result is 74. What was Byron's original choice for the integer N?
4. It is currently 2:00. After how many minutes will the hour and minute hands on a 12-hour clock form a $45°$ angle between them for the first time? Express your answer as a common fraction.
5. A video on the video sharing site CyberCast has 100 more likes than dislikes. The *score* of a video is computed as $1.2l - 0.9d$, where l and d are the numbers of likes and dislikes, respectively. If the score of the video is 285, and each vote is either a like or a dislike, compute the total number of votes on the video.
   (a) 600
   (b) 900
   (c) 1050
   (d) 1200
   (e) none of the above
6. The sum of the squares of the first n positive integers ends in 6. Compute the smallest possible value of n.
7. What is the average value of the base-9 positive integers less than 100_9, when they are read as base-10 integers?
8. The letters in CYBERMATH are each written down on their own slip of paper and tossed into a bag. Three slips of paper are then drawn out of the bag, without replacement. Compute the probability that at least two vowels (A, E, I, O, U, or Y) are drawn. Express your answer as a common fraction.
9. For every positive integer n, let $Q(n)$ be the closest integer to $\sqrt{n}$. What is the value of $Q(1) + Q(2) + Q(3) + \cdots + Q(100)$?
10. If x and y are positive real numbers so that $x^2 + y^2 = 74$ and $x + y = 12$, compute $x^3 + y^3$.
11. Suppose f is a quadratic polynomial with $f(0) = 1$, $f(1) = 20$, and $f(2) + f(3) = 22$. Then the coefficient of its linear term can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
12. Let m and n be integers for which n is a multiple of m and $\frac{mn}{m+n} = \frac{10}{3}$. Compute the sum of the possible values of mn.
13. Line segment $\overline{AB}$ has side length 5, and point C lies in the plane of $\overline{AB}$ with $AC^2 = CB^2 + 1$. Point M is the midpoint of $\overline{AB}$. If $MC = 5$, the square of the distance from C to $\overline{AB}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
14. Let a and b be real numbers for which $\frac{a^2+b^2}{ab} = 7$ and $a + b = 6$. Compute $\max(a, b)$. Express your answer in simplest radical form.

15. Square $ABCD$ has side length 1. A semicircle has arc AB and center at the midpoint of $\overline{AB}$. Point Q lies on the circumference of the semicircle so that the distance from Q to $\overline{CD}$ is 1.2. If Q is closer to C than to D, the value of QC^2 can be written in the form $\frac{m-\sqrt{n}}{p}$, where m, n, and p are positive integers, and n is not a multiple of the square of any prime. Compute $m+n+p$.

16. A magical tetrahedral die initially has 1 red face and 3 blue faces, each equally likely to be rolled. Whenever a red face comes up, one of the blue faces changes to a red face. Compute the expected number of rolls before all faces of the die turn red.

17. What is the area of a triangle whose side lengths are $\sqrt{2}$, 3, and $\sqrt{11+2\sqrt{3}+2\sqrt{6}}$? Express your answer as a common fraction in simplest radical form.

18. The sum of two positive integer perfect squares ends in the digits 34. Compute the second-smallest possible sum of the square roots of the integers.

19. Suppose that

$$\sum_{n=2}^{10} \frac{n^2}{\binom{n}{2}\binom{n+1}{2}} = \frac{p}{q}$$

for relatively prime positive integers p and q. Compute $p+q$.

20. A set S of positive integers is said to be *relatively open* if the set $T := \{s_1 + s_2 : s_1, s_2 \in S, s_1 \neq s_2\}$ is missing at least half of the positive integers from $\min(S)$ to $\max(S)$, inclusive. For example, $S = \{1, 2, 3, 10\}$ is relatively open, because $T = \{3, 4, 5, 11, 12, 13\}$ is missing 1 and 6-10 among $[1, 10]$, whereas $S = \{1, 2, 3, 5, 6, 10\}$ would not be relatively open, because $T = \{3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16\}$ is only missing 1, 2, and 10 among $[1, 10]$. Compute the number of nonempty subsets of $\{1, 2, 3, 4, 5, 6\}$ that are relatively open.