cyber math

pen Tournament

nline

Contest File (Europe/Africa)

High School Division

Saturday, March 26, 2022

1. For how many positive integers n will repeatedly subtracting n from 1000 eventually produce 1?
   (a) 4
   (b) 6
   (c) 8
   (d) 9
   (e) none of the above

2. A rectangle has side lengths in the ratio 3:4. If the diagonal has length d, and the rectangle's area is 1, compute d. Express your answer as a common fraction in simplest radical form.

3. A cafe serves milk in five flavors: plain, strawberry, chocolate, vanilla, and caramel. Mikhail wants to drink milk at the cafe for five consecutive days, but does not want to drink the same flavor for two days in a row, nor does he want to drink plain milk on the last day or chocolate milk on the first day. In how many ways can he decide which milk flavors to drink on each of the five days?
   (a) 625
   (b) 768
   (c) 1024
   (d) 1278
   (e) none of the above

4. Compute the sum of the coordinates of all rational points (points with rational coordinates) that lie on the circle $(x-8)^2+(y-6)^2=100$.
   (a) 168
   (b) 140
   (c) 112
   (d) 56
   (e) none of the above

5. At least 4 percent of the positive integers between n and $2n$, inclusive, are perfect squares. Compute the largest possible value of the positive integer n.

6. Suppose that the real part of $z=(x+iy)^3$ is 259 for positive integers x and y, where $i=\sqrt{-1}$. Compute the imaginary part of z.

7. A base-b positive integer N whose base-10 value is 83 is written on a blackboard. When its rightmost digit is erased, its base-10 value becomes 16. When N is read as a base-10 number, what is $N+b$?

8. Rectangle $ABCD$ has $AB=2$ and $BC=1$. Point E lies on $\overline{BC}$ such that the extension of $\overline{AE}$ past E intersects the extension of $\overline{CD}$ at F and the area of $\triangle EFD$ is 2022. Compute BE.

9. Three fair six-sided dice, each labeled with the integers from 1 through 6 inclusive, are rolled. Compute the probability that no pair of dice rolls has a product that is a perfect square, but the product of all three dice rolls is a perfect square.

10. For each positive integer n, let $Z(n)$ be the set of all positions (from left to right) of the instances of the digit 0 in n. For example, $Z(2022)=\{2\}$, and $Z(102030405)=\{2,4,6,8\}$. Compute

$$\sum_{n=1}^{2022}\sum_{d\in Z(n)} d.$$

11. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$, with M lying on $\overline{BC}$ so that $\overline{AM}$ bisects $m\angle BAC$. Point D lies on $\overline{AC}$ so that $\overline{BD}$ and $\overline{AM}$ intersect at point K. If the distance from K to $\overline{BC}$ is 4, then $\frac{AD}{DC} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

12. Let x and y be distinct real numbers such that $x^2 + 7x + 11 = y^2 + 7y + 11 = 0$. Compute

$$\frac{x^2}{x^2y + y^3} + \frac{y^2}{x^3 + xy^2}.$$

13. Delilah wants to buy exactly 18 donuts at MoreDonuts. The available flavors are vanilla, strawberry, and chocolate cream, but station 1 only serves vanilla and strawberry, station 2 only serves vanilla and chocolate cream, and station 3 only serves strawberry and chocolate cream. Suppose Delilah buys exactly 6 donuts from each station. In how many ways can she buy donuts? (Two ways are different if there are a different number of any given type of donut.)

14. For each positive integer n, let $R(n)$ be the ratio of the number of even divisors of n to the number of divisors of n. Compute the fractional part of $R(1) + R(2) + R(3) + \cdots + R(100)$.

15. For each positive integer $n = p_1^{j_1} p_2^{j_2} p_3^{j_3} \cdots p_k^{j_k}$, where the p_i are distinct prime numbers and the j_i are positive integers, define

$$f(n) := \prod_{i=1}^{k} (p_i - 1).$$

Let $\varphi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the largest positive integer N such that

$$\sum_{n=1}^{N} \frac{f(n)}{\varphi(n)} < 10.$$

16. Compute the number of ordered pairs (a, b) of positive integers with $1 \le a, b \le 100$ such that the system of equations $x - y = a$, $x^3 - y^3 = b$ has a real solution (x, y).

17. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Point P lies on $\overline{BC}$ such that the circumcircle Ω of $\triangle APC$ intersects $\overline{AB}$ at $Q \neq A$ and $AQ = \frac{13}{4}$. Then $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

18. A 3×3 grid of squares, filled in with 5 X's and 4 O's, is called *achievable* if it corresponds to some sequence of nine alternating X and O placements on the grid, beginning and ending with X's, such that at no point before the last square is filled in does the grid contain three X's or O's in a single row, column, or diagonal. For each achievable 3×3 tic-tac-toe grid, define its *connectedness* as the total number of squares in contiguous blocks of X's and O's that are not singletons. For example, the following tic-tac-toe grid is achievable and has connectedness 8:

X	X	O
X	O	O
X	O	X

Compute the expected connectedness of an achievable 3×3 grid with 5 X's and 4 O's chosen uniformly at random.

19. Triangle ABC has $AB = 3$, $BC = 4$, and $CA = 5$. A circle O centered at the incenter I of $\triangle ABC$ has radius $r \in (\sqrt{2}, \sqrt{5})$ so that the points D, E, F, and G of intersection of O with $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, and $\overline{CA}$, respectively, are vertices of pentagon $DBEFG$ with area 5. Then $r^2 = \frac{m - n\sqrt{p}}{q}$, where m, n, p, and q are positive integers with $\gcd(m, n, q) = 1$ and p square-free. Compute $m + n + p + q$.

20. Giorgio takes a walk along a 4×4 square grid, beginning at the top-left square. At each square, he chooses to walk to each of the squares adjacent to him with equal probability, but he can only walk down and to the right. Once he reaches the lower-right corner of the grid, he stops. Giorgio's walk is then interpreted as a 4×4 matrix, with all squares that Giorgio has visited (including his starting point) being 1s and all squares that Giorgio has never visited being 0s.

    Define the *anti-Dyck-ness* $D(A)$ of the resulting matrix A as follows. Let B be the 3×3 matrix whose elements on or below the diagonal are exactly the same as those *strictly* below the diagonal of A, and $B = B^t$. (Here we denote by B^t the transpose of B; i.e. the matrix whose rows are the columns of B and whose columns are the rows of B, obtained by flipping B over its diagonal.) We say that $D(A)$ is equal to the determinant of B.

    For example, if Giorgio takes 2 steps down, 1 step right, 1 step down, and 2 steps right, we will have
    $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$
    and thus,
    $$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$
    In this case, $D(A) = \det(B) = -1$.

    Compute the probability that $D(A) = 0$.