cyber math

pen Tournament

nline

Contest File (Asia/Australia)

High School Division

Saturday, March 26, 2022

1. Suppose that $(\log_2(x))^2 + \log_4(x) = 68$. Compute the product of the possible real values of x. Express your answer as a common fraction in simplest radical form.

2. Siya starts counting backward from 95 in increments of 4. As soon as she gets to a negative number, she starts counting upward in increments of 5. As soon as she reaches a number above 100, she counts backwards in increments of 6, and so forth. What is the 100^{th} number Siya counts?

   (a) 51
   (b) 54
   (c) 65
   (d) 69
   (e) none of the above

3. Let $MATH$ be a unit square. Point P lies on $\overline{AT}$ with $MP = \frac{5}{4}$, and point Q is the intersection of the bisector of $m\angle MAT$ with $\overline{MP}$. Compute the area of triangle MAQ. Express your answer as a common fraction.

4. What is the value of

$$\sqrt{\frac{7+\sqrt{37}}{2}} + \sqrt{\frac{7-\sqrt{37}}{2}},$$

   in simplest radical form?

5. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Point D is the foot of the perpendicular from A to $\overline{BC}$. Square $DEFG$ is drawn inside triangle ADC, with E on $\overline{AD}$, F on $\overline{AC}$, and G on $\overline{BC}$. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

6. Jules wants to compute the value of $14_b \cdot 62_b$ in an integer base $b \geq 7$, but accidentally computes $41_b \cdot 26_b$ instead. For each positive integer $b \geq 7$, let $f(b)$ be the positive difference between the incorrect value and the correct value. Compute the remainder when $f(7)+f(8)+f(9)+f(10)+\cdots+f(100)$ is divided by 1000.

   (a) 96
   (b) 259
   (c) 447
   (d) 676
   (e) none of the above

7. For how many ordered pairs (x, y) of positive integers with $x, y \leq 100$ is $xy + 2x + 3y$ a multiple of 6?

8. Triangle PQR has $PQ = 3$, $QR = 4$, $RP = 5$, and a right angle at Q. Let S lie on $\overline{PR}$ so that $\overline{QS}$ is an altitude of $\triangle PQR$. Compute the smallest possible distance between two points lying on the inscribed circles of $\triangle PQS$ and $\triangle SQR$.

9. Perpendicular lines ℓ_1 and ℓ_2, whose y-intercepts sum to 22, intersect at the point $(5, -4)$. The largest possible value for one of the two y-intercepts can be written in the form $p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of a prime. Compute $p + q + r$.

10. Leatrice writes down a 3-digit base-5 positive integer, chosen uniformly at random. Nora then chooses one of the digits in Leatrice's number uniformly at random and erases it. Compute the probability that the resulting number is a valid base-3 number.

11. What is the units digit of

$$\binom{2022}{1} + \binom{2021}{2} + \binom{2020}{3} + \binom{2019}{4} + \cdots + \binom{1013}{1010} + \binom{1012}{1011}?$$

12. On moving day, Suri is packing five cubical boxes (each of which is long enough to a side to contain all smaller boxes) into the moving truck. A larger box can contain all smaller boxes, except the smallest box must be nested in some other, larger box, since it has fragile contents. In how many ways can she pack the boxes into the truck?

13. In triangle ABC, $AB = 6$, $BC = 8$, and $CA = 10$. Points P and Q lie on $\overline{AB}$ and $\overline{BC}$, respectively, with $PB = BQ = 2$, so that $\triangle APQ$ has circumcircle O which intersects $\overline{CA}$ at point R. Compute $\frac{AR}{AC}$.

14. N fair coins are flipped. If the expected product of the numbers of heads and tails flipped is an integer, compute the largest possible value of N less than or equal to 2022.

15. Triangle ABC in the xy-plane has $AB = 3$, $BC = 4$, and $CA = 5$, with $A = (0, 3)$, $B = (0, 0)$, and $C = (4, 0)$. Point I is the incenter of $\triangle ABC$. The smallest possible area of a rectangle with side lengths parallel to the coordinate axes circumscribing the circumcircles of $\triangle ABI$, $\triangle BCI$, and $\triangle CAI$ can be written in the form $\frac{m+n\sqrt{p}+q\sqrt{r}+s\sqrt{t}}{u}$, where all of the variables are positive integers, $\gcd(m, n, q, s) = 1$, and none of p, r, or t are divisible by the square of a prime number. Compute $m + n + p + q + r + s + t + u$.

16. A fair six-sided die, labeled with the integers from 1 to 6 inclusive on its faces, is rolled until a 6 comes up for the first time. After the first roll of a 6, let $E(N)$ be the expected ratio of the product of all rolls up to that point to the quantity $N^{\text{total number of rolls } -1}$. If $E(N) = 2022$, then $N = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

17. Let $f(x)$ be a monic cubic polynomial with real coefficients whose roots sum to 9. Suppose that the reciprocals of the roots of $g(x) := f(x)^2$ sum to $-\frac{1}{3}$. Compute the coefficient of the linear term of $f(x)$ for which the value of $f(1) + g(1)$ attains its minimum. Express your answer as a common fraction.

18. Suppose that $(a_i)_{i\geq 1}$ is a sequence such that $a_1 = 2$, $a_2 = 6$, $a_3 = 24$, and for all $n \geq 4$, $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$. Compute the sum of all positive integers $n \leq 1000$ for which $a_n - 8$ is a power of 2.

19. Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let M be the midpoint of $\overline{BC}$, and let E be the projection of C onto the extension of $\overline{AM}$. The circumcircle of $\triangle ACE$ intersects $\overline{AB}$ at point $F \neq A$. Compute $\frac{BF}{BA}$. Express your answer as a common fraction.

20. Suppose that z_1, z_2, and z_3 are complex numbers that are the roots of some cubic polynomial with positive integer coefficients and leading coefficient 1 such that

$$\frac{z_1+1}{z_1^2} + \frac{z_2+1}{z_2^2} + \frac{z_3+1}{z_3^2} = -\frac{21}{16}$$

and

$$z_1^2 + z_2^2 + z_3^2 = 73.$$

Compute the smallest possible value of $|z_1^3 + z_2^3 + z_3^3|$.