Online Open Tournament

Contest File (North/Central/South America)

High School Division

Saturday, March 26, 2022

1. What is the sum of all positive integers that evenly divide 1680 but not 240?

2. The product of 9 consecutive positive integers, the median of which is 7, is equal to t times the product of 5 consecutive positive integers, the median of which is 7. Compute t.

   (a) 280
   (b) 720
   (c) 1320
   (d) 2520
   (e) none of the above

3. Let ABC be a triangle with $AB = 20$, $BC = 22$, and $CA^2 = 444$. Compute the area of $\triangle ABC$. Express your answer in simplest radical form.

4. Square $ABCD$ has side length 20, and square $CEFG$ has side length 22, with $\overline{BC}$ lying in the interior of $\overline{CG}$. Compute the area of pentagon $ABGFD$.

5. A problem on a test has an answer that is a common fraction, and contains the instruction, "Let your answer be of the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$." If the raw answer were some positive real quantity r, the final answer would be some integer N, but if the raw answer were instead $r+\frac{1}{5}$, the final answer would be 5. Compute the sum of all possible values of N.

   (a) 75
   (b) 90
   (c) 110
   (d) 135
   (e) none of the above

6. Consider the set $S = \{1, 2, 3, \cdots, 1100\}$. What is the fewest number of elements that we must remove from S so that there is no pair of distinct elements in S that sum to 2022?

7. Compute the number of positive integers $n \leq 10000$ for which there lies a perfect square between n and $n + 100$, inclusive.

8. Given that $0 \leq x < 2\pi$ is a real number for which
$$\frac{\sin(3x)}{\sin(x)} = \frac{3}{4},$$
the value of $\cos^2(x)$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

9. Compute
$$\sum_{k=0}^{10} k\binom{10}{k}^2.$$

10. Regular hexagon $ABCDEF$ has side length 1. How many of the $\binom{6}{4} = 15$ quadrilaterals whose vertices are four distinct vertices of the hexagon have an inscribed circle?

11. Triangle ABC has $AB = 20$, $BC = 22$, and $CA = 21$. Point D lies on $\overline{AC}$ such that $m\angle ABD = m\angle CBD$. Point F lies on $\overline{AB}$ such that $\overline{FD} \parallel \overline{BC}$. The area of $\triangle BDF$ can be written in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that $\gcd(p, r) = 1$ and q is square-free. Compute $p+q+r$.

12. Triangle ABC has $AB = 5$, $BC = 12$. For some real numbers $x \in [0, 1]$ and t, if $\cos(m\angle ABC) = x$, then $CA^2 = t$, but if $\sin(m\angle ABC) = x$, then $CA^2 = t - 24$. Compute t.

13. Define the *A-index* of a permutation of a string of letters to be the total number of A's in contiguous substrings of the string that have length at least 2. For example, the A-index of the permutation AAALBMA of ALABAMA is 3, and the A-index of the permutation ABRCAADBARA of ABRACADABRA is 2. Compute the sum of all A-indices of all permutations of the string AAAABBCC.

14. Triangle ABC has $AB = 3$, $BC = 4$, and $CA = 5$. Point P lies on $\overline{BC}$ so that $\tan(m\angle BAP) + \tan(m\angle PAC) = 1$. Compute AP^2.

15. Suppose that z_1 and z_2 are complex numbers with $|z_1| = |z_2| = 1$ and $z_1 + z_2 = 1 + \frac{3}{2}i$. Compute the product of the imaginary parts of z_1 and z_2.

16. Evaluate the sum
$$\sum_{n=1}^{\infty} \frac{n^4}{n!}.$$

17. For each real number x, define
$$S(x) := ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots$$
If $0 \le x < 2\pi$ is a real number such that $S(x)^{12} = 1$, compute the product of the possible values of x.

18. Equilateral triangle ABC has side length 2. When $\triangle ABC$ is rotated by $45°$ clockwise about vertex A to map vertices A, B, and C to A', B', and C', respectively, in triangle $A'B'C'$ (where $A' = A$), the area of overlap between triangles ABC and $A'B'C'$ can be written in the form $\frac{p+\sqrt{q}-\sqrt{r}-\sqrt{s}}{t}$, where p, q, r, s, and t are positive integers. Compute $p + q + r + s + t$.

19. For how many ordered pairs (a, b) of positive integers with $1 \le a, b \le 10$ does the equation $4x^4 + 2ax^3 + bx^2 + ax + 1 = 0$ have exactly two real solutions in x (up to multiplicity)?

20. For each real number $x \neq 0$, define
$$f(x) := \frac{x+1}{x},$$
and define
$$g(x, h) := \frac{f(x+h) - f(x)}{h}.$$
For some integer c, there exist exactly two distinct real values of x such that $g(x, c+1) - g(x, c) = x$. Compute the sum of the squares of the possible values of c.