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Contest Solutions

Invitational Math Tournament (High School)

Saturday, April 2, 2022

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Qualifying Round

1. How many positive integer second powers less than or equal to 10^6 are also fourth powers?

   Answer: $\boxed{31}$

   Such a second power must be the square of a perfect square. Since $10^6 = (10^3)^2$, and there are 31 perfect squares less than or equal to $10^3 = 1000$, there are $\boxed{31}$ such second powers.

2. A rectangular prism has integer dimensions x, y, and z, surface area 120, and volume 72. If $x^2y + y^2z + z^2x = 312$, compute $x + y + z$.

   Answer: $\boxed{14}$

   We have $xy + yz + zx = 60$ and $xyz = 72$, so $\frac{xy+yz+zx}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{6}$. Noting that $3 \cdot \frac{1}{4} < \frac{5}{6}$, at least one of x, y, and z must be 2 or 3. Thus, (x, y, z) can be a permutation of $(3, 4, 4)$, $(3, 3, 6)$, or $(2, 6, 6)$. Only $(2, 6, 6)$ satisfies $xyz = 72$, so $x + y + z = \boxed{14}$.

3. How many permutations $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ have the property that $a_1 + a_2 + a_3$ and $a_3 + a_4 + a_5$ differ by at most 2?

   Answer: $\boxed{72}$

   The positive difference between the two sums is equal to $|(a_1 + a_2) - (a_4 + a_5)|$. We can perform casework on the value of a_3; if $a_3 = 1$, then $a_1 + a_2 + a_4 + a_5 = 14$ and $a_1 + a_2$ must be 6, 7, or 8. We have $(a_1, a_2) \in \{(2, 4), (2, 5), (3, 4), (3, 5)\}$ in this case. If $a_3 = 2$, then, likewise, $a_1 + a_2$ must be 6 or 7, and $(a_1, a_2) \in \{(1, 5), (2, 4), (2, 5), (3, 4)\}$. If $a_3 = 3$, then $a_1 + a_2$ must be 5, 6, or 7, and $(a_1, a_2) \in \{(1, 4), (1, 5), (2, 4), (2, 5)\}$. If $a_3 = 4$, we have $a_1 + a_2 \in \{5, 6\}$, and $(a_1, a_2) \in \{(1, 5), (2, 3)\}$. Finally, if $a_3 = 5$, we have $a_1 + a_2 \in \{4, 5, 6\}$, from which we get $(a_1, a_2) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. Altogether, we have $2!^2 \cdot 18 = \boxed{72}$ permutations with the desired property (where $2!^2 = 4$ is the number of ways to order the blocks (a_1, a_2) and (a_4, a_5)).

4. Let $S(n)$ denote the sum of the digits of the integer n. Suppose that $S(a) = 5$, $S(b) = 7$, and $S(100a + b) \neq 12$. Compute the number of possible ordered pairs (a, b) with $a, b \leq 1000$.

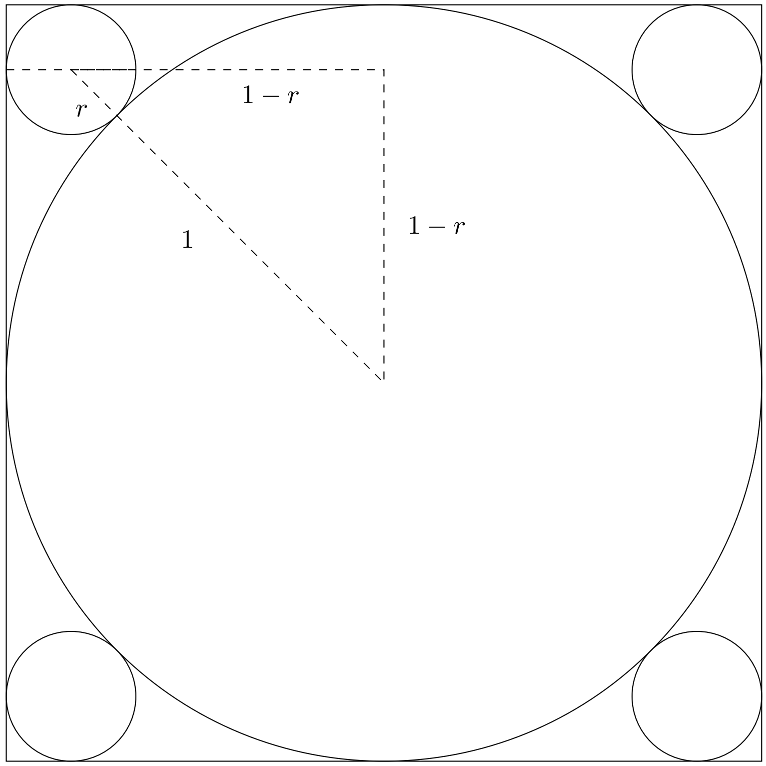
   Answer: $\boxed{15}$

   First note that if $b \leq 100$, adding b to $100a$ (which ends in two zeros) will certainly result in a digit sum of 12, as a and b will merely be concatenated. Thus, $b \in \{106, 115, 124, \cdots, 160, 205, \cdots, 601, 610, 700\}$. Now consider the units digit of a. The only way to avoid $100a + b$ having a digit sum of 12 is for there to be a thousands-digit carryover, which motivates us to consider the units digit of a and the hundreds digit of b. If $a = 5$, b can be 502-520, 601-610, or 700. If a ends in 4 (is 14 or 104), b can be 601, 610, or 700. If a ends in 3 (is 23, 113, or 203), then b can only be 700. If a ends in 2, 1, or 0, then no choice of b can cause this carryover. Altogether, we end up with $\boxed{15}$ choices for the ordered pair (a, b).

5. Square $ABCD$ has side length 2. Circle O shares its center with that of $ABCD$ and has radius 1. Circles O_1, O_2, O_3, and O_4 are tangent to circle O and the pairs $\overline{AB}$ and $\overline{AD}$, $\overline{AB}$ and $\overline{BC}$, $\overline{BC}$ and $\overline{CD}$, and $\overline{CD}$ and $\overline{DA}$ of side lengths of the square, respectively. Compute the area of the square whose vertices are the centers of O_1, O_2, O_3, and O_4. Express your answer in simplest radical form.

   Answer: $\boxed{48 - 32\sqrt{2}}$

We draw the following diagram:



from which $(r+1)^2 = 2(1-r)^2 \implies r = 3 - 2\sqrt{2}$. The side length of the square is $2 - 2r$, so its area is $4(1-r)^2 = 4(2\sqrt{2}-2)^2 = \boxed{48 - 32\sqrt{2}}$.

6. Triangle ABC has $AB^2 = 37$, $BC^2 = 61$, and $CA^2 = 49$. Compute the area of $\triangle ABC$. Express your answer as a common fraction in simplest radical form.

Answer: $\boxed{\frac{47\sqrt{3}}{4}}$

Consider point P in the interior of $\triangle ABC$ with $AP = 3$, $BP = 4$, and $CP = 5$ such that $m\angle APB = m\angle BPC = m\angle CPA = 120°$. By the law of cosines, we can verify that $\triangle ABC$ indeed has the claimed side lengths. By the area formula $\frac{ab \sin C}{2}$, we get an area of $\frac{47}{2}\sin(120°) = \boxed{\frac{47\sqrt{3}}{4}}$ for $\triangle ABC$.

7. For some positive integers b and c, the polynomial $x^3 + 4x^2 - bx + c$ has three integer roots. Given that $|b - c| \leq 10$, compute the sum of all possible values of $|b - c|$.

Answer: $\boxed{9}$

Exactly one of the roots must be negative, since their sum is -4, their second symmetric sum is negative, and the product is also negative. With $(-6, 1, 1)$, we have $b = 11$ and $c = 6$; with $(-7, 1, 2)$, we have $b = 19$ and $c = 14$; and so forth, where all permutations of triples of the form $(-a, 1, a-5)$ give $c = b - 5$. For $(-8, 2, 2)$, we have $b = 28$ and $c = 32$; for $(-9, 2, 3)$, we have $b = 39$ and $c = 54$, which is a bad triple. Indeed, suppose one root is $p \leq -9$ and the others sum to $-4 - p$ and are not 1. Then their product is at most $p \cdot 2 \cdot (-6 - p) = -12p - 2p^2$, and their second symmetric sum is at least $2p + p(-6-p) + 2(-6-p) = -p^2 - 6p - 12$, so that $|b - c| \geq p^2 + 6p - 12$. For $p \leq -9$, this is always at least 11. Hence, the only possible values of $|b - c|$ that are at most 10 are 5 and 4, which sum to $\boxed{9}$.

8. Let r, s, and t be the roots of the polynomial $4x^3 - 22x^2 + 36x - c$ for some positive integer c. If r, s, and t are the side lengths of a triangle with positive area, the area of the circumcircle of the triangle can be written in the form $\frac{p}{q}\pi$, where p and q are relatively prime positive integers. Find $p + q$.

   Answer: $\boxed{368}$

   Only for $c = 15$ do r, s, t satisfy the triangle inequality; for $c \leq 14$, two of the roots are nonreal, and for $c \geq 16$, the roots are 2 and $\frac{7 \pm \sqrt{17}}{4}$. Already these do not satisfy the triangle inequality; as c increases, the product rst increases with $r + s + t$ remaining constant. Thus, one root must increase while the other decreases, and this will not allow (r, s, t) to satisfy the triangle inequality. Hence $c = 15$, and by the rational root theorem, the roots are $\frac{5}{2}$ and $\frac{3 \pm \sqrt{3}}{2}$. Thus, $R = \frac{15}{\sqrt{143}}$ by a routine computation, meaning that the circumcircle's area is $\frac{225}{143}\pi$, and $p + q = \boxed{368}$. (To get the intuition for this without calculus, first try plugging in $x = 1$ as a potential root to get $c = 18$. We want to locally minimize the value of $4x^3 - 22x^2 + 36x$ over $x \geq 0$, and plotting this on a graphing calculator reveals that the local minimum is just below 15. With calculus, one can show that the minimum is exactly $\frac{451 - 13\sqrt{13}}{27}$.)

9. Triangle ABC has $AB = 7$, $BC = 8$, and $CA = 9$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively, with $AD = AE$. Given that the area of $\triangle ADE$ is 1, compute DE^2. Express your answer in simplest radical form.

   Answer: $\boxed{\sqrt{5}}$

   Let $AD = AE = d$; by the law of cosines with $\cos(m\angle BAC) = \frac{11}{21}$ (from the cosine addition identity, writing $m\angle BAC = m\angle BAD + m\angle DAC$, where D is the foot of the altitude from A to $\overline{BC}$), we get

$$DE^2 = d^2 + d^2 - 2(d)(d)(\cos(m\angle BAC)) = \frac{20}{21}d^2, \tag{1}$$

   so $DE = \frac{2\sqrt{105}}{21}d$. The side lengths of $\triangle ADE$ are then d, d, and $\frac{2\sqrt{105}}{21}d$, so its area is d^2 times that of a triangle with side lengths 1, 1, and $\frac{2\sqrt{105}}{21}$. This can be easily calculated to be $\frac{4\sqrt{5}}{21}$ (say, by dropping an altitude), so $d^2 = \frac{21}{4\sqrt{5}}$ and $DE^2 = \frac{20}{21} \cdot \frac{21}{4\sqrt{5}} = \boxed{\sqrt{5}}$.

10. Compute the sum of the coefficients of the monic polynomial of minimal degree with integer coefficients which has $3^{\frac{1}{5}} + 3^{\frac{4}{5}}$ as a root.

   Answer: $\boxed{-53}$

   Let $a = \sqrt[5]{3}$ and $b = \sqrt[5]{81}$ (with $ab = 3$), so that $a + b$ is the root. Notice that

$$(a + b)^5 = a^5 + b^5 + (a + b)(5ab((a + b)^2 - ab)) \tag{2}$$

   (upon writing $a^2 + ab + b^2$ as $(a + b)^2 - ab$)

$$= 84 + (a + b)(15((a + b)^2 - 3)) \tag{3}$$

$$= 84 + (a + b)(15(a + b)^2 - 45) \tag{4}$$

$$= 84 + 15(a + b)^3 - 9(a + b). \tag{5}$$

   Letting $c = a + b$, we get $c^5 = 84 + 15c^3 - 45c$, or $c^5 - 15c^3 + 45c - 84 = 0$. As this polynomial is monic and irreducible (since all other roots are non-real, by Descartes' law of signs), the desired sum is $\boxed{-53}$.

A note on Live Round scoring

Each of the 3 problems is scored out of 10 points, for a maximum of 30 points. The following is a rough guideline for the assignment of scores:

- 10 points: Perfect solution.
- 9 points: Extremely minor computational error (sign error, addition error, etc).
- 8 points: Mostly correct, but with a few minor computational errors or a minor mis-application of a formula or idea.
- 7 points: Has the structure of a correct proof, but slightly sloppy or imprecise (although not incorrect) in the execution.
- 6 points: Has the general structure of a correct proof, but the execution is slightly sloppy and handwavy; in addition, there may be a few missing key components.
- 5 points: Half-complete; usually one part of a problem is done correctly but not another, or the student has forgotten a critical component of the proof (a good example is showing minimality/maximality but not achievability).
- 4 points: Possible misapplication of a critical idea, but on the right track.
- 3 points: A considerable amount of nontrivial progress that has the potential to lead to a solution with significant work.
- 2 points: Some nontrivial progress.
- 1 point: Some tangential observations related to the problem.
- 0 points: No or only entirely trivial progress. **An answer (even if correct) with no justification should be scored zero.**

There is some built-in leeway here, and scores are ultimately assigned at each judge's personal discretion.

Live Round

1. What is the maximum value of $a\sin(x) + b\cos(x)$ over all real numbers x, in terms of positive real numbers a and b?

   Answer: $\boxed{\sqrt{a^2+b^2}}$

   Solution: By the Cauchy-Schwarz inequality,

   $$(a\sin(x) + b\cos(x))^2 \le (a^2+b^2)(\sin^2(x)+\cos^2(x)) = a^2+b^2 \tag{6}$$

   so that the maximum value is $\sqrt{a^2+b^2}$. Note that this is actually attainable; equality holds when $\frac{\sin(x)}{a} = \frac{\cos(x)}{b}$, or $\frac{\sin(x)}{\cos(x)} = \tan(x) = \frac{a}{b}$; since $a, b > 0$, $\tan(x) > 0$, and tan has range $[0, \infty)$ for $0 \le x < \frac{\pi}{2}$.

   Scoring guidelines:

   - A valiant, but handwavy and non-rigorous attempt with an *incorrect answer* should receive either 0/10 or 1/10 points, depending on whether the attempt contains ideas that are potentially applicable to the problem (at judge's discretion).
   - A valiant attempt with a correct answer, but a very handwavy and non-rigorous proof that does not use Cauchy-Schwarz, should receive between 1/10 and 4/10 points (at the judge's discretion).

- If the student attempts to apply Cauchy-Schwarz: minimum of 2/10 points.
- If the student demonstrates a clear understanding of the Cauchy-Schwarz inequality in general: minimum of 4/10 points.
- Unclear explanation of fundamental steps: no more than 6/10 points.
- Showing maximality, but not achievability: maximum of 7/10 points.
- Mistake in the achievability step: deduct 1 or 2 points.
- Silly mistake (e.g. a^2+b^2 instead of $\sqrt{a^2+b^2}$), with everything else correct and clearly explained: deduct 1 point.

2. Alpha and Beta each have N dollars. They flip a fair coin together, and if it is heads, Alpha gives a dollar to Beta; if it is tails, Beta gives a dollar to Alpha. They stop flipping when one of them goes bankrupt and the other has $2N$ dollars. What is the expected number of times that they will end up flipping the coin?

   Answer: $\boxed{N^2}$

   Solution: An equivalent problem is to consider the expected number of flips before Alpha goes bankrupt (since Beta has $2N$ dollars if and only if Alpha has zero, and vice versa). By the theory of Markov chains, we can represent the expected number of flips before bankruptcy in terms of Alpha's current wealth n as $E(n)$, where $E(0) = 0$ and $E(n) = \frac{1}{2}E(n-1) + \frac{1}{2}E(n+1) + 1$ for all $n \geq 1$. (This is because, with probability $\frac{1}{2}$, Alpha's wealth will decrease to $n-1$, and it then will take $E(n-1)$ flips, on average, to go bankrupt; likewise for $n+1$; but then we must add the 1 flip it took for Alpha's wealth to change.) Rewriting this as $\frac{1}{2}E(n+1) - E(n) + \frac{1}{2}E(n-1) = -1$, we may guess the solution form as $an^2 + bn + c$, so that

$$\frac{1}{2}(a(n+1)^2 + b(n+1) + c) - (an^2 + bn + c) + \frac{1}{2}(a(n-1)^2 + b(n-1) + c) = -1 \tag{7}$$

$$\implies \frac{1}{2}(an^2 + (2a+b)n + a + b + c) - (an^2 + bn + c) + \frac{1}{2}(an^2 + (b-2a)n + a - b + c) = -1 \tag{8}$$

   so that $a = -1$ and $E(n) = -n^2 + bn + c$. Since $E(0) = 0$, we have $c = 0$, and since $E(2N) = 0$ as well, we have $-4n^2 + 2nb = 0$, or $b = 2n$. It follows that $E(n) = n^2$ for all n, so in particular, the expected number of flips for Alpha to go bankrupt from a starting wealth of n dollars is $\boxed{n^2}$.

   Scoring guidelines:

   - A valiant, but handwavy and non-rigorous attempt with an *incorrect answer* should receive either 0/10 or 1/10 points, depending on whether the attempt contains ideas that are potentially applicable to the problem (at judge's discretion).
   - A valiant attempt with a correct answer, but a very handwavy and non-rigorous proof, should receive between 1/10 and 4/10 points (at the judge's discretion).
   - Identifying the general/hand-wavy recursive aspect of the problem, but failing to make more significant progress, is worth 1/10 points.
   - Writing an incorrect recursive equation, but with some general/hand-wavy understanding of what else might need to be done to solve the problem from there, is worth 2/10 points.
   - Writing *and justifying* the correct recursive equation: minimum of 2/10 points.
   - Demonstrating some general understanding of how to solve a non-homogeneous recurrence relation: minimum of 2/10 points. In conjunction with the above item: a solution that obtains the correct recursion and makes a valiant attempt at a solution with some promising ideas should receive no fewer than 4/10 points.
   - Guessing the correct form of the solution with justification: minimum of 6/10 points.

- General interpretations about the problem based on intuition, but with slightly unrigorous proof, should receive between 4/10 and 8/10 points, at the judge's discretion.
- Correctly solving the recurrence in general, but forgetting boundary conditions, is worth 8/10 points.
- Finishing the problem correctly is worth the remaining 2 points.
- Silly mistake, with everything else correct and clearly explained: deduct 1 point.

3. For each positive integer k, define

$$S_k := \sum_{n=1}^{\infty} \frac{n^k}{n!}.$$

Prove that S_k is e times the k^{th} Bell number $B(k)$, where $B(k)$ is the number of ways of placing k labeled balls into k indistinguishable bins.

Solution: We have

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{n^k}{n!} && (9)\\ &= \sum_{n=1}^{\infty} \frac{n^{k-1}}{(n-1)!} && (10)\\ &= \sum_{n=0}^{\infty} \frac{(n+1)^{k-1}}{n!} && (11)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{k-1} \binom{k-1}{j} n^j && (12)\\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{n=0}^{\infty} \frac{n^j}{n!} && (13)\\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} S_j. && (14) \end{aligned}$$

Finally, observe that equation (14) expresses S_k in terms of all smaller S_j; by induction, if S_k indeed were e times the k^{th} Bell number, we'd have an additional $(k+1)^{st}$ ball to place in any of the k indistinguishable bins. The number of such placements is equal to the sum of the binomial coefficients $\binom{k}{j}$, since it is also the number of partitions of k. As $S_1 = e$ (well-known), we are done.

Scoring guidelines:

- A valiant, but handwavy and non-rigorous attempt, should receive at most 2/10 points, depending on the extent to which the attempt contains ideas that are potentially applicable to the problem (at judge's discretion).
- General understanding of how to re-index and rewrite the sum (i.e. reaching equation (11)) is worth at least 3/10 points.
- Concluding equation (12) from equation (11): minimum of 7/10 points. **Note that this step is the crux of the problem!**
- Reaching equation (14): minimum of 8/10 points.
- A correct combinatorial interpretation of (14) is worth the remaining 2 points.
- Demonstrating some general understanding of how the combinatorial interpretation *might* work, but without a rigorous proof of said interpreation, should receive somewhere between 2/10 and 4/10 points, at the judge's discretion.
- Forgetting the factor of e (from the base case): deduct 2 points.
- Silly mistake, with everything else correct and clearly explained: deduct 1 point.