



pen Tournament

Contest File

Invitational Math Competition (High School)

Saturday, April 2, 2022

## Qualifying Round (High School)

1. How many positive integer second powers less than or equal to  $10^6$  are also fourth powers?
2. A rectangular prism has integer dimensions  $x$ ,  $y$ , and  $z$ , surface area 120, and volume 72. If  $x^2y + y^2z + z^2x = 312$ , compute  $x + y + z$ .
3. How many permutations  $(a_1, a_2, a_3, a_4, a_5)$  of  $(1, 2, 3, 4, 5)$  have the property that  $a_1 + a_2 + a_3$  and  $a_3 + a_4 + a_5$  differ by at most 2?
4. Let  $S(n)$  denote the sum of the digits of the integer  $n$ . Suppose that  $S(a) = 5$ ,  $S(b) = 7$ , and  $S(100a + b) \neq 12$ . Compute the number of possible ordered pairs  $(a, b)$  with  $a, b \leq 1000$ .
5. Square  $ABCD$  has side length 2. Circle  $O$  shares its center with that of  $ABCD$  and has radius 1. Circles  $O_1, O_2, O_3$ , and  $O_4$  are tangent to circle  $O$  and the pairs  $\overline{AB}$  and  $\overline{AD}$ ,  $\overline{AB}$  and  $\overline{BC}$ ,  $\overline{BC}$  and  $\overline{CD}$ , and  $\overline{CD}$  and  $\overline{DA}$  of side lengths of the square, respectively. Compute the area of the square whose vertices are the centers of  $O_1, O_2, O_3$ , and  $O_4$ . Express your answer in simplest radical form.
6. Triangle  $ABC$  has  $AB^2 = 37$ ,  $BC^2 = 61$ , and  $CA^2 = 49$ . Compute the area of  $\triangle ABC$ . Express your answer as a common fraction in simplest radical form.
7. For some positive integers  $b$  and  $c$ , the polynomial  $x^3 + 4x^2 - bx + c$  has three integer roots. Given that  $|b - c| \leq 10$ , compute the sum of all possible values of  $|b - c|$ .
8. Let  $r, s$ , and  $t$  be the roots of the polynomial  $4x^3 - 22x^2 + 36x - c$  for some positive integer  $c$ . If  $r, s$ , and  $t$  are the side lengths of a triangle with positive area, the area of the circumcircle of the triangle can be written in the form  $\frac{p}{q}\pi$ , where  $p$  and  $q$  are relatively prime positive integers. Find  $p + q$ .
9. Triangle  $ABC$  has  $AB = 7$ ,  $BC = 8$ , and  $CA = 9$ . Points  $D$  and  $E$  lie on  $\overline{AB}$  and  $\overline{AC}$ , respectively, with  $AD = AE$ . Given that the area of  $\triangle ADE$  is 1, compute  $DE^2$ . Express your answer in simplest radical form.
10. Compute the sum of the coefficients of the monic polynomial of minimal degree with integer coefficients which has  $3^{\frac{1}{5}} + 3^{\frac{4}{5}}$  as a root.

## Live Round (High School)

1. What is the maximum value of  $a \sin(x) + b \cos(x)$  over all real numbers  $x$ , in terms of positive real numbers  $a$  and  $b$ ?
2. Alpha and Beta each have  $N$  dollars. They flip a fair coin together, and if it is heads, Alpha gives a dollar to Beta; if it is tails, Beta gives a dollar to Alpha. They stop flipping when one of them goes bankrupt and the other has  $2N$  dollars. What is the expected number of times that they will end up flipping the coin?
3. For each positive integer  $k$ , define

$$S_k := \sum_{n=1}^{\infty} \frac{n^k}{n!}.$$

Prove that  $S_k$  is  $e$  times the  $k^{\text{th}}$  Bell number  $B(k)$ , where  $B(k)$  is the number of ways of placing  $k$  labeled balls into  $k$  indistinguishable bins.