Contest Solutions

Invitational Math Tournament (Middle School)

Saturday, April 2, 2022

The problems and solutions for this competition were prepared by the teaching staff of CyberMath Academy. Thanks to Thinula de Silva and Freya Edholm for reviewing these problems and solutions!
1. How many dots are in a formation with 1000 rows, where rows 1 and 1000 consist of a single dot, rows 2 and 999 consist of 3 dots, rows 3 and 998 consists of 5 dots, and rows 4-997 each consist of 7 dots?

Answer: 6976

This is a $7 \cdot 1000$ array with triangular formations of $1 + 2 + 3 = 6$ dots removed at each of the four corners, hence consisting of $7000 - 24 = 6976$ dots.

2. What is the sum of all positive integers $x$ satisfying $|x(9 - x)| < 72$?

Answer: 105

From $-72 < x(9 - x) < 72$, we get $x \leq 14$, so the requested sum is $\frac{14 \cdot 15}{2} = 105$.

3. How many positive integers have digits summing to 5 and no digits of zero?

Answer: 16

We can inductively show that there are $2^{s-1}$ such numbers for a sum of $s \leq 9$, hence 16 numbers with $s = 5$. (This is because we can place a 1 either before the string of sum $s-1$, or after it.) It is also not difficult to count the 16 possibilities directly.

4. Four fair six-sided dice, each labeled with the positive integers from 1 through 6 inclusive, are rolled. What is the probability that no two distinct dice come up with numbers whose product is a perfect square? Express your answer as a common fraction.

Answer: $\frac{1}{6}$

If any two dice come up with the same number, those dice will multiply to a perfect square. Thus, we certainly must have all four dice rolls be distinct (but this is not in itself a sufficient condition). Note that $(1, 4)$ is also a bad pair, but there are no others where the rolls are different. Excluding the $(1, 4) = 6$ choices that have both 1 and 4 from the $(3) = 15$ choices of four distinct dice rolls, we get 9 good roll choices and their 4! permutations. Altogether, the probability of rolling a good 4-tuple is $\frac{9 \cdot 4!}{6^4} = \frac{1}{6}$.

5. For how many positive integers $n \leq 100$ is $n(n+1)^2(n+2)$ divisible by 88?

Answer: 20

One of $n$, $n + 1$, or $n + 2$ must be a multiple of 11, so $n \equiv 0, 9, 10 \mod 11$. In addition, we must have three factors of 2; if $n$ is even, then $n$ and $n + 1$ are even while $n + 1$ is odd, so one of them must be a multiple of 4 (but this is guaranteed, so all even $n$ congruent to 0, 9, 10 mod 11 work). If $n$ is odd, on the other hand, $n + 1$ is the only even number among $n$, $n + 1$, and $n + 2$, so $n + 1$ must be a multiple of 4 (in order for $(n + 1)^2$ to be a multiple of 8 as needed). This implies $n \equiv 3 \mod 4$ in the odd case. Altogether, in the case where $n$ is even and congruent to 0, 9, 10 mod 11, we get 3 solutions in every block of 22 by the Chinese remainder theorem, hence 12 solutions up to $n = 88$, and also the solution $n = 98$, for 13 solutions. In the second case, where $n \equiv 3 \mod 4$ and also congruent to 0, 9, 10 mod 11, we get 3 solutions in every block of 44 by CRT, hence 6 solutions up to and including $n = 888$. We have $n = 99$ as a solution as well, so there are a total of $13 + 7 = 20$ possible values for $n$.

6. Regular hexagon $ABCDEF$ has side length 3. Points $G, H, I, J, K,$ and $L$ lie on $AB, BC, CD, DE, EF,$ and $FA$ respectively such that $AG = BH = CI = DJ = EK = FL = 1$. Compute the area of
7. Label the positions of the letters in the word CYBERMATH with the positive integers from 1 to 9, inclusive, from left to right. Let $S$ be the sum of the products of the left-to-right positions of A, B, and C, respectively, over all 9! permutations of CYBERMATH. Compute $S \mod 10000$.

Answer: \[4000\]

Note that $S$ will be $6! \cdot 3! = 4320$ times the sum over all \(\binom{9}{3}\) unique choices of the positions of A, B, and C, without respect to order. This sum is equal to \(1(2 \cdot 42 + 3 \cdot 39 + 4 \cdot 35 + 5 \cdot 30 + 6 \cdot 24 + 7 \cdot 17 + 8 \cdot 9) + 2(3 \cdot 39 + 4 \cdot 35 + 5 \cdot 30 + 6 \cdot 24 + 7 \cdot 17 + 8 \cdot 9) + 3(4 \cdot 35 + \cdots + 8 \cdot 9) + \cdots + 7 \cdot 8 \cdot 9\), where we continually drop the first term of the sum; through some calculation, we get that this is 826 + 1484 + 1875 + 1940 + 1675 + 1146 + 504 = 9450. Multiplying by 4320 gives \(S \equiv 4000 \mod 10000\).

8. Triangle $ABC$ has the property that the length of $AC$ is an integer. Point $D$ lies on $AC$ with $m\angle ABD = m\angle DBC$. If the area of $\triangle ABD$ is 15 and the area of $\triangle BCD$ is 30, compute the smallest possible length of $AC$.

Answer: 12

By the angle bisector theorem, we know that $\frac{AB}{BC} = \frac{15}{30} = \frac{1}{2}$, so we can let $AB = x$ and $BC = 2x$. Since the area of $\triangle ABC$ is 45, the length of the altitude from $A$ to foot $F \in BC$ of the perpendicular to $BC$ is $\frac{45}{2}$. Thus, $BF = \sqrt{x^2 - \frac{2025}{x^2}}$, and so $FC = 2x - \sqrt{x^2 - \frac{2025}{x^2}}$. Then $AC = \sqrt{2025 + \left(2x - \sqrt{x^2 - \frac{2025}{x^2}}\right)^2} = \sqrt{2025 + 4x^2 - 4\sqrt{x^2 - \frac{2025}{x^2}} + \left(x^2 - \frac{2025}{x^2}\right)} = \sqrt{5x^2 - 4\sqrt{x^2 - \frac{2025}{x^2}}}.$

Therefore, we want to minimize $5x^2 - 4\sqrt{x^2 - \frac{2025}{x^2}}$, while also making sure it is a perfect square. Set it equal to $m$, and we want the smallest possible value of $m$; this yields $m - 5x^2 = 4\sqrt{x^2 - \frac{2025}{x^2}}$, or $16x^4 - 180m^2 = m^2 - 10mx^2 + 25x^4 \implies 9x^2 - 10mx^2 + (m^2 - 180) = 0$. For $m$ to be a minimum, we want the discriminant of this quadratic to be zero, so that $100m^2 = 36(m^2 - 180^2) \implies m = 135$. Therefore, $AC \geq \sqrt{135}$, and the smallest possible integer value of $AC$ is then 12.

9. A positive integer $n$ is called cube-special if there exist distinct positive integers $a$ and $b$ with $a^3 - na = b^3 - nb$. How many positive integers less than or equal to 100 are cube-special.

Answer: 22

We have $a^3 - b^3 = na - nb = n(a - b)$; since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we have $n = a^2 + ab + b^2$. For $b - a = 1$, we get $n = a^2 + a(a + 1) + (a + 1)^2 = 3a^2 + 3a + 1$, so $n \in \{7, 19, 37, 61, 91\}$ in order to be less than or equal to 100 and cube-special in this case. For $b - a = 2$, we similarly get $n = 3a^2 + 6a + 4$, so that $n \in \{13, 28, 47, 76\}$. For $b - a = 3$, $n = 3a^2 + 9a + 9$, so $n \in \{21, 39, 63, 93\}$. For $b - a = 4$, $n \in \{31, 52, 79\}$; for $b - a = 5$, $n \in \{43, 67, 97\}$; for $b - a = 6$, $n \in \{57, 84\}$; for $b - a = 7$, $n \in \{73\}$; and for $b - a = 8$, $n \in \{91\}$. For $k := b - a \geq 9$, we’d have $n = 3a^2 + 3ka + k^2$, which for $a = 1$ evaluates to $3 + 3k + k^2$, but this is more than 100 for $k \geq 9$. Thus, \{7, 13, 19, 21, 28, 31, 37, 39, 43, 49, 52, 57, 61, 63, 67, 73, 76, 79, 84, 91, 93, 97\} is the set of all cube-special numbers up to 100, of which there are 22.
10. How many permutations of \((1, 2, 3, 4, 5, 6, 7, 8)\) do not have any two consecutive positive integers in increasing order and in adjacent positions? For example, \((2, 1, 4, 3)\) is one such permutation of \((1, 2, 3, 4)\), but \((3, 1, 2, 4)\) is not.

Answer: \(16687\)

Let \(P(n)\) be this number of permutations of \((1, 2, 3, \cdots, n)\). We define \(P(n)\) recursively, with base cases \(P(1) = P(2) = 1\). For \(n \geq 3\), we may form a desired permutation of \(P(n)\) either by placing the number \(n\) between two consecutive numbers in a permutation of length \(n - 2\) with \textit{exactly} one pair of consecutive numbers (which can be done in \(n - 2\) ways), or in \(n - 1\) possible positions (i.e. not immediately to the right of \(n - 1\)) given a desired permutation of length \(n - 1\). This gives rise to the recurrence \(P(n) = (n - 1)P(n - 1) + (n - 2)P(n - 2)\), from which \(P(8) = 16687\).

A note on Live Round scoring

Each of the 3 problems is scored out of 10 points, for a maximum of 30 points. The following is a rough guideline for the assignment of scores:

- 10 points: Perfect solution.
- 9 points: Extremely minor computational error (sign error, addition error, etc).
- 8 points: Mostly correct, but with a few minor computational errors or a minor mis-application of a formula or idea.
- 7 points: Has the structure of a correct proof, but slightly sloppy or imprecise (although not incorrect) in the execution.
- 6 points: Has the general structure of a correct proof, but the execution is slightly sloppy and handwavy; in addition, there may be a few missing key components.
- 5 points: Half-complete; usually one part of a problem is done correctly but not another, or the student has forgotten a critical component of the proof (a good example is showing minimality/maximality but not achievability).
- 4 points: Possible misapplication of a critical idea, but on the right track.
- 3 points: A considerable amount of nontrivial progress that has the potential to lead to a solution with significant work.
- 2 points: Some nontrivial progress.
- 1 point: Some tangential observations related to the problem.
- 0 points: No or only entirely trivial progress. \textbf{An answer (even if correct) with no justification should be scored zero.}

There is some built-in leeway here, and scores are ultimately assigned at each judge’s personal discretion.

Live Round

1. Let rectangle \(ABCD\) have \(AB = 2\). Point \(E \neq C, D\) lies on \(CD\) such that \(\triangle AED\) is similar to \(\triangle ABE\). Show that \(E\) must be the midpoint of \(CD\).

Let \(BC = u\) and \(ED = x\); then from \(\frac{AE}{AB} = \frac{ED}{EB} = \frac{DA}{EA}\), we get that \(\frac{\sqrt{x^2 + u^2}}{2} = \frac{x}{\sqrt{(2-x)^2 + u^2}} = \frac{u}{\sqrt{x^2 + u^2}}\).

From \(\frac{\sqrt{x^2 + u^2}}{2} = \frac{u}{\sqrt{x^2 + u^2}}\), it follows that \(x^2 + u^2 = 2u\), or \(x = \sqrt{2u - u^2}\). Then \(\frac{\sqrt{2u}}{2} = \frac{\sqrt{2u - u^2}}{\sqrt{(2-\sqrt{2u - u^2})^2 + u^2}}\).
so we get that $\frac{u}{2} = \frac{-2u - u^2}{4 + 2u - u^2 + 2u} \Rightarrow 2 - 2\sqrt{2u - u^2} + u = 2 - u \Rightarrow u = \sqrt{2u - u^2} \Rightarrow u = 0, 1.$

Since $E \neq D$, $ED = u \neq 0$, so $u = 1$ and $E$ is the midpoint of $\overline{CD}$.

Scoring guidelines:

- A valiant attempt, but one that does not employ the correct ideas of similarity, should receive either 0/10 or 1/10 points, depending on whether the attempt contains other ideas that are potentially applicable to the problem (at judge’s discretion).
- If the student correctly applies similarity ratios: minimum of 2/10 points.
- Reaching $x = \sqrt{2u - u^2}$ is worth 5/10 points.
- Solving the equations correctly is worth 9/10 points if the explanation is entirely thorough and clear.
- Unclear explanation of equation solving: deduct points as appropriate, perhaps 2-3 points from the total score. (There is some leeway with this item.)
- Forgetting about $u \neq 0$ is a 1-point deduction.
- Other silly mistake (e.g. $a^2 + b^2$ instead of $\sqrt{a^2 + b^2}$), with everything else correct and clearly explained: deduct 1 point.

2. Let $f(x) = ax^3 + bx^2 + cx + d$ be a quadratic polynomial with positive integer coefficients. Prove that the sum of the squares of the roots of $f(x)$ does not depend on $d$. If this sum is 1, what is the smallest possible value of $f(1)$?

Answer: $9$

Let the roots be $r$, $s$, and $t$; then $r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + rt + st) = \left(\frac{b}{a}\right)^2 - 2\frac{c}{a}$ by Vieta’s formulas, which is independent of $d$. For $\frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2} = 1$, we have $b^2 - 2ac = a^2$. If $a = 1$, we get $b^2 - 2c = 1$, implying that $b$ is odd, so $b \geq 3$. Then $c = 4$, and $f(1) = a + b + c + d \geq 9$ (since $d \geq 1$, being a positive integer). On the other hand, if $a \geq 2$ and is even, from $b^2 = a^2 + 2ac$, we’d get that $b$ is even, hence $b \geq 4$ (since $b = 2$ implies $a^2 + 2ac = 4$, which is impossible with $a = 2$). So then $a^2 + 2ac \geq 16$; with $a = 2$, this gives $c \geq 3$, or $a + b + c + d \geq 10$, and with $a \geq 4$, this gives $a + b + c + d \geq 4 + 4 + 1 + 1 = 10$. With $a \geq 3$ odd, $b \geq 4$, since $3^2 + 2 \cdot 3 \cdot 1 = 15$. Thus, $a + b + c + d \geq 9$, and 9 is indeed minimal.

Scoring guidelines:

- A valiant, but handwavy and non-rigorous attempt with an incorrect answer should receive either 0/10 or 1/10 points, depending on whether the attempt contains ideas that are potentially applicable to the problem (at judge’s discretion).
- A valiant attempt with a correct answer, but a very handwavy and non-rigorous proof, should receive between 1/10 and 3/10 points (at the judge’s discretion).
- Writing the correct factorization of $r^2 + s^2 + t^2$ is worth a minimum of 2/10 points.
- Correctly applying Vieta’s formulas to complete the first part of the problem is worth 4/10 points.
- Given the previous item: a general casework approach on $a$, with not much success, is worth 5/10 points.
- Identifying $f(1) = a + b + c + d$ is worth +1 point on its own.
- Using a mod 4 argument in the casework is worth a minimum of 3 points (on top of the correct use of Vieta’s formulas, this is a minimum of 7/10 points).
- Identifying 9 as achievable, but not attempting a minimality argument, is worth at most 6/10 points.
- Silly mistake, with everything else correct and clearly explained: deduct 1 point.
3. Show that, for each positive integer $1 \leq z \leq 8$, the numbers of 10-digit positive integers whose digits sum to 10 with exactly $z$ zeros and $9 - z$ zeros are equal.

The $z$ zeros can go in 9 places (anything but the leftmost digit), and $\binom{9}{z} = \binom{9}{9-z}$. We have $10 - z$ nonzero digits summing to 10 in the case that we have $z$ zeros, and $1 + z$ nonzero digits summing to 10 in the case that we have $9 - z$ zeros, or $10 - z$ digits summing to $z$ as opposed to $1 + z$ digits summing to $9 - z$. By stars-and-bars, we have $\binom{(10-z)+z-1}{(10-z)-1} = \binom{9}{9-z}$ ways in the first case, and $\binom{(1+z)+(9-z)-1}{(1+z)-1} = \binom{9}{z}$ ways in the second case; these are the same for all $1 \leq z \leq 8$.

Scoring guidelines:

- A valiant, but handwavy and non-rigorous attempt, should receive at most 2/10 points, depending on the extent to which the attempt contains ideas that are potentially applicable to the problem (at judge’s discretion).
- Observing that we have equal numbers of ways to place the zeros is worth a minimum of 2/10 points.
- General understanding of a sketch of how stars-and-bars might be broadly applicable is worth 4-5 points, at the judge’s discretion.
- Rigorously applying stars-and-bars is worth 10/10 points, before deductions.
- Silly mistake, with everything else correct and clearly explained: deduct 1 point.