AMC Lecture Notes

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Contents

§1 Systems of Equations

1.1 Substitution

Example 1.1. Solve the system below for x and y:

$$x - y = 4, \quad 2x + y = 29.$$

Solution. From the first equation, we see $x = y + 4$. Substituting into the second equation,

$$2(y + 4) + y = 29 \implies 3y + 8 = 29 \implies y = 7.$$

Therefore, $x = 11$ and the ordered pair of solutions is $(x, y) = \boxed{(11, 7)}$. □

Example 1.2. Solve the following system of equations, where x and y are real numbers:

$$x + y + \sqrt{x + y} = 30, \quad x - y + \sqrt{x - y} = 12.$$

Solution. Let $u = \sqrt{x + y}$ and $v = \sqrt{x - y}$ in the equations. Therefore,

$$u^2 + u = 30, \quad v^2 + v = 12.$$

The first quadratic has solutions $u = 5$ and $u = -6$ and the second $v = 3$ and $v = -4$. However, the principal square root is positive, so $\sqrt{x + y} = u = 5$ and $\sqrt{x - y} = v = 3$. Hence,

$$\begin{cases} x + y &= 25 \\ x - y &= 9 \end{cases} \implies (x, y) = \boxed{(17, 8)}.$$ □

Example 1.3. Find the positive integer x such that $x(x + 1)(x + 2)(x + 3) + 1 = 271^2$.

Solution. We substitute $y = x^2 + 3x + 1$ and multiply the inner and outer terms:

$$\begin{aligned} x(x + 1)(x + 2)(x + 3) + 1 &= \left(x^2 + 3x\right)\left(x^2 + 3x + 2\right) + 1 \\ &= (y - 1)(y + 1) + 1 \\ &= y^2. \end{aligned}$$

Therefore, $y = 271$. We factor $x^2 + 3x + 1 = 271$ as $(x + 18)(x - 15) = 0$, so $x = \boxed{15}$. □

Exercises

1.1. (Mandelbrot) Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is. How many years from now will Gene be five times as old as Carol?

1.2. Find all values of x such that $\sqrt{4x - 3} + \dfrac{10}{\sqrt{4x - 3}} = 7$.

1.3. (ARML) Find all values of r such that $\left(r^2 + 5r\right)\left(r^2 + 5r + 3\right) = 4$.

1.4$\star$ (HMMT) Find all real numbers x and y with $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$.

1.2 Elimination

Example 1.4. Solve the following system of equations using elimination:

$$x + 2y = 18, \quad 7x - 6y = 26.$$

Solution. We multiply the first equation by 3 to eliminate y:

$$3x + 6y = 54, \quad 7x - 6y = 26.$$

Adding these equations gives $10x = 80$, so $x = 8$. Then $y = 5$, so the solution is $(x, y) = (8, 5)$. □

Example 1.5. (COMC) Find all ordered pairs (x, y) that satisfy both $xy^2 = 10^8$ and $x^3/y = 10^{10}$.

Solution. Squaring the second equation gives $x^6/y^2 = 10^{20}$. Multiplying by the first equation:

$$\left(xy^2\right)\left(\frac{x^6}{y^2}\right) = \left(10^8\right)\left(10^{20}\right).$$

Therefore $x^7 = 10^{28}$, so $x = 10^4 = 10000$. Substituting this into the second equation, $x^3/y = 10^{10}$, we find $y = 10^2 = 100$, so the solution pair is $(x, y) = \boxed{(10000, 100)}$. □

Example 1.6. Find all ordered pairs (x, y) such that $3x^2 - 2xy = 336$ and $2y^2 - 3xy = 84$.

Solution. We begin by factoring out x and y from the two equations:

$$x\,(3x - 2y) = 336, \quad y\,(2y - 3x) = 84.$$

Observe that $3x - 2y = -\,(2y - 3x)$ and since they are nonzero, we divide the two equations:

$$\frac{x(3x - 2y)}{y(2y - 3x)} = \frac{-x}{y} = \frac{336}{84} = 4.$$

Hence, $x = -4y$. We substitute this into the first equation to arrive at

$$3\,(-4y)^2 - 2\,(-4y)\,y = 56y^2 = 336 \implies y^2 = 6.$$

Hence, $y = \pm\sqrt{6}$. The corresponding solution pairs are $(x, y) = \boxed{(-4\sqrt{6}, \sqrt{6}), (4\sqrt{6}, -\sqrt{6})}$. □

Exercises

1.5. (AMC 12) When a bucket is two-thirds full of water, the bucket and water weigh x kilograms. When the bucket is one-half full of water the total weight is y kilograms. In terms of x and y, what is the total weight in kilograms when the bucket is full of water?

1.6. Find all ordered pairs (x, y) such that $\sqrt{x} + \sqrt{y} = 7$ and $3\sqrt{x} - 4\sqrt{y} = -14$.

1.7. (Mandelbrot) Let x and y be real numbers satisfying $\dfrac{2}{x} = \dfrac{y}{3} = \dfrac{x}{y}$. Determine x^3.

1.8. Find all r and s such that $r^2 - 2rs = 16$ and $rs - 2s^2 = 4$.

1.3 Symmetry

Example 1.7. Solve the system of equations

$$2x + y + z = 32, \quad x + 2y + z = 3, \quad x + y + 2z = 21.$$

Solution. Since the equations are symmetric, we add them:

$$4x + 4y + 4z = 56 \implies x + y + z = 14.$$

Subtracting this from the three equations, we see $(x, y, z) = \boxed{(18, -11, 7)}$. □

Example 1.8. Solve the system of equations $xy = 27$, $xz = 8$, and $yz = 6$.

Solution. Multiplying the three equations, we see

$$(xy)(xz)(yz) = x^2y^2z^2 = 27 \cdot 8 \cdot 6 = 36^2.$$

Therefore, $xyz = 36$. Dividing, we see $(x, y, z) = (36/6, 36/8, 36/27) = \boxed{(6, 9/2, 4/3)}$. □

Example 1.9 (AMC 12)**.** Find xyz if x, y, and z are positive reals satisfying the system

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad z + \frac{1}{x} = 7/3.$$

Solution. We want to find the product xyz, therefore, we multiply the three equations:

$$\begin{aligned}\left(x + \frac{1}{y}\right)\left(y + \frac{1}{z}\right)\left(z + \frac{1}{x}\right) &= xyz + (x + y + z) + \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \frac{1}{xyz} \\ &= 4 \cdot 1 \cdot \frac{7}{3} = \frac{28}{3}.\end{aligned}$$

Adding the equations, $\left(x + \frac{1}{y}\right) + \left(y + \frac{1}{z}\right) + \left(z + \frac{1}{x}\right) = 4 + 1 + \frac{7}{3} = \frac{22}{3}$. Therefore,

$$xyz + \frac{22}{3} + \frac{1}{xyz} = \frac{28}{3} \implies xyz + \frac{1}{xyz} = 2 \implies xyz = \boxed{1}.$$ □

Exercises

1.9. Find all solutions (x, y, z) to the system of equations

$$\begin{aligned}(x + y)(x + y + z) &= 15, \\ (y + z)(x + y + z) &= 6, \\ (z + x)(x + y + z) &= -3.\end{aligned}$$

1.10. Find $ab + \frac{1}{ab}$ if a and b are positive reals such that $a + \frac{1}{b} = 6$ and $b + \frac{1}{a} = 4$.

1.4 Challenge Problems

1.11. Solve the following system of equations in w, x, y, z:

$$\begin{aligned} 3w + x + y + z &= 20 \\ w + 3x + y + z &= 6 \\ w + x + 3y + z &= 44 \\ w + x + y + 3z &= 2. \end{aligned}$$

1.12. (AIME) Find the positive solution to

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0.$$

1.13. (Purple Comet) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \ \frac{bc}{b+c} = 4, \text{ and } \frac{ca}{c+a} = 5.$$

Compute the value of $\dfrac{abc}{ab + bc + ca}$.

1.14. (Math League) If $x < y$, find the ordered pair of real numbers (x, y) which satisfies

$$x^3 + y^3 = 400, \quad x^2y + xy^2 = 200.$$

1.15. (Purple Comet) Let x, y, z be positive real numbers satisfying the simultaneous equations

$$\begin{aligned} x(y^2 + yz + z^2) &= 3y + 10z \\ y(z^2 + zx + x^2) &= 21z + 24x \\ z(x^2 + xy + y^2) &= 7x + 28y. \end{aligned}$$

Find $xy + yz + zx$.

1.16. Find the four values of x that satisfy $(x - 3)^4 + (x - 5)^4 = -8$.

1.5 Solutions

Exercises for Section 1.1

1.1 Let the current ages of Gene and Carol be g and c, respectively. Two years ago, Gene was $g-2$ years old and Carol was $c-2$ years old. Therefore, we have the system of equations

$$g = 7c, \quad g - 2 = 9(c-2).$$

We substitute the first equation into the second:

$$7c - 2 = 9(c-2) \implies 7c - 2 = 9c - 18 \implies 2c = 16 \implies c = 8.$$

Therefore, $g = 7 \cdot 8 = 56$. Let the amount of time until Gene is five tims as old as Carol be t, so

$$56 + t = 5(8+t) \implies 56 + t = 40 + 5t \implies t = \boxed{4 \text{ years}}.$$

1.2 We substitute $u = \sqrt{4x-3}$ into the equation to give

$$u + \frac{10}{u} = 7 \implies u^2 - 7u + 10 = 0 \implies (u-2)(u-5) = 0.$$

If $u = 2$, then $\sqrt{4x-3} = 2$, so $x = \boxed{7/4}$. If $u = 5$, then $\sqrt{4x-3} = 5$, so $4x - 3 = 25$ and $x = \boxed{7}$.

1.3 We substitute $s = r^2 + 5r + 1.5$ into the equation:

$$(s - 1.5)(s + 1.5) = 4 \implies s^2 - 2.25 = 4 \implies s = \pm 2.5$$

We solve the quadratics $r^2 + 5r + 1.5 = 2.5$ and $r^2 + 5r + 1.5 = -2.5$. The latter factors as $(r+4)(r+1) = 0$ with roots $r = \boxed{-4}$ and $r = \boxed{-1}$. For the former, we use the quadratic formula:

$$r^2 + 5r - 1 = 0 \implies r = \frac{-5 \pm \sqrt{25 - 4(-1)}}{2} = \boxed{\frac{-5+\sqrt{29}}{2}, \frac{-5-\sqrt{29}}{2}}.$$

1.4 We substitute $u = xy$ and $v = x + y$ into the system of equations to obtain:

$$u^2 + v^2 = 40, \quad u + v = 8.$$

Squaring the latter equation, we have $u^2 + 2uv + v^2 = 64$, therefore $uv = 12$. The solutions are $(u, v) = (6, 2), (2, 6)$. Then x and y are the roots of $z^2 - 2z + 6$ or $z^2 - 6z + 2$. However, for the first quadratic, $\Delta = 2^2 - 4 \cdot 6 = -20 < 0$, therefore, there are no real roots. In the second case,

$$z^2 - 6z + 2 = 0 \implies (z-3)^2 = 7 \implies z = 3 \pm \sqrt{7}.$$

Therefore, the solution pairs are $(x, y) = \boxed{(3+\sqrt{7}, 3-\sqrt{7}), (3-\sqrt{7}, 3+\sqrt{7})}$.

Exercises for Section 1.2

1.5 Let the weight of the bucket without any water be b and the weight of the water be w:

$$b + 2/3w = x, \quad b + 1/2w = y.$$

Subtracting shows $1/6w = x - y \implies w = 6x - 6y$. Substituting this into the second equation,

$$b + 1/2\,(6x - 6y) = y \implies b + 3x - 3y = y \implies b = 4y - 3x.$$

The total weight when the bucket is full of water is $b + w = (4y - 3x) + (6x - 6y) = \boxed{3x - 2y}$.

1.6 Adding 4 times the first equation to the second, we see

$$4\left(\sqrt{x} + \sqrt{y}\right) + \left(3\sqrt{x} - 4\sqrt{y}\right) = 7\sqrt{x} = 4 \cdot 7 - 14 = 14.$$

Therefore, $\sqrt{x} = 2$, so $x = 4$. Substituting this back into the first equation, we have $\sqrt{y} = 5$, so $y = 25$ and the only pair of solutions is $(x, y) = \boxed{(4, 25)}$.

1.7 We are given $\dfrac{2}{x} = \dfrac{y}{3}$ and $\dfrac{2}{x} = \dfrac{x}{y}$. Multiplying these equations gives

$$\frac{4}{x^2} = \frac{x}{3} \implies x^3 = \boxed{12}.$$

1.8 We rewrite the first equation as $r(r - 2s) = 16$, and the second equation as $s(r - 2s) = 4$. Since they are nonzero, dividing gives $r/s = 16/4$, hence $r = 4s$. Substituting into the first equation gives $4s^2 - 2s^2 = 4$, so $s = \sqrt{2}$ or $s = -\sqrt{2}$. Therefore, $(r, s) = \boxed{(4\sqrt{2}, \sqrt{2}), (-4\sqrt{2}, -\sqrt{2})}$.

Exercises for Section 1.3

1.9 Adding the three equations, $(2x + 2y + 2z)\,(x + y + z) = 18 \implies x + y + z = \pm 3$.

- If $x + y + z = 3$, then the first equation tells us $(3 - z)\,(3) = 15$, so $z = -2$. Similarly, $(3 - x)\,(3) = 6$, so $x = 1$ and $(3 - y)\,(3) = -3$, so $y = 4$. Hence, $(x, y, z) = \mathbf{(1, 4, -2)}$.
- If $x + y + z = -3$, then we have $(-3 - z)\,(-3) = 15$, so $z = 2$. Similarly, $(-3 - x)\,(-3) = 6$, so $x = -1$ and $(-3 - y)\,(-3) = -3$, so $y = -4$. Hence, $(x, y, z) = \mathbf{(-1, -4, 2)}$.

In summary, the solutions to the system are $(x, y, z) = \boxed{(1, 4, -2), (-1, -4, 2)}$.

1.10 Multiplying the equations $a + \frac{1}{b} = 6$ and $b + \frac{1}{a} = 4$,

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = 24 \implies ab + \frac{1}{ab} = \boxed{22}.$$